

AD-A174 503

HIGH FREQUENCY ANALYSIS OF CIRCULAR ARCHES(U) APPLIED

1/2

RESEARCH ASSOCIATES INC ALBUQUERQUE NM

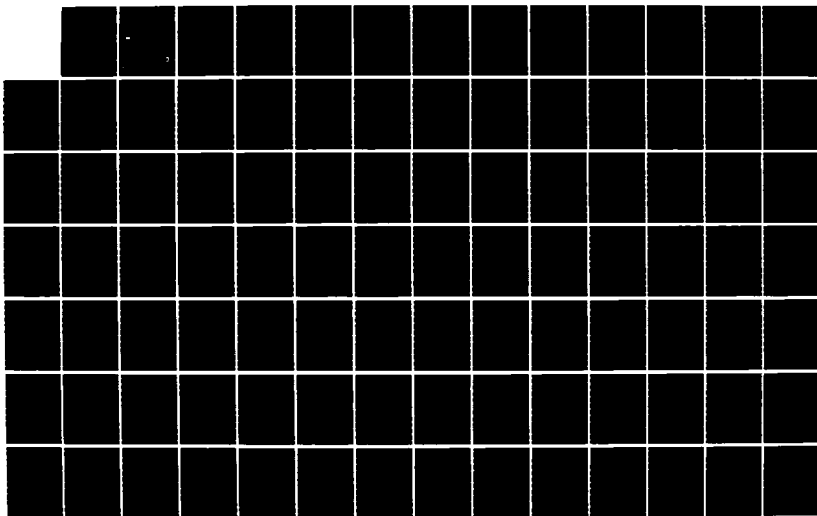
D H MERKLE ET AL. SEP 86 5956 AFML-TR-86-22

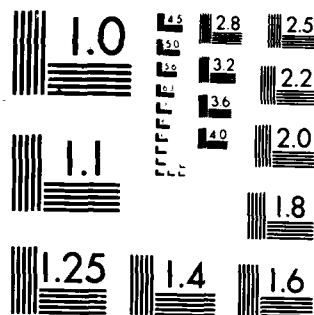
UNCLASSIFIED

F29601-85-C-0029

F/G 13/13

NL

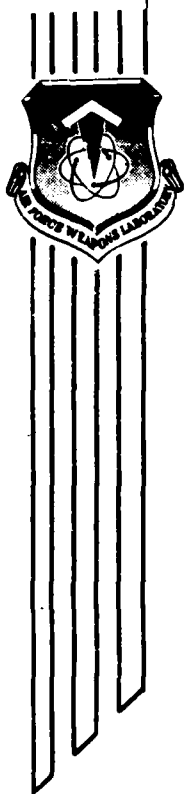




MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A



AD-A174 503



## HIGH FREQUENCY ANALYSIS OF CIRCULAR ARCHES

Douglas H. Merkle  
Laurence D. Merkle

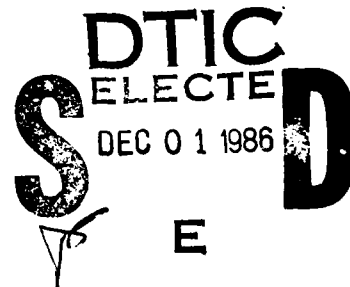
Applied Research Associates, Inc.  
4300 San Mateo, NE  
Albuquerque, New Mexico 87110

September 1986

Final Report

Approved for public release; distribution unlimited.

AIR FORCE WEAPONS LABORATORY  
Air Force Systems Command  
Kirtland Air Force Base, NM 87117-6008



NTIC FILE COPY

86 12 01 024

This final report was prepared by Applied Research Associates, Inc., Albuquerque, New Mexico, under Contract F29601-85-C-0029, Job Order 88091395, with the Air Force Weapons Laboratory, Kirtland Air Force Base, New Mexico. Dr Timothy J. Ross (NTES) was the Laboratory Project Officer-in-Charge.

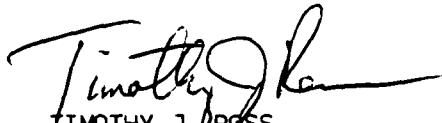
When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely Government-related procurement, the United States Government incurs no responsibility or any obligation whatsoever. The fact that the Government may have formulated or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication, or otherwise in any manner construed, as licensing the holder, or any other person or corporation; or as conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

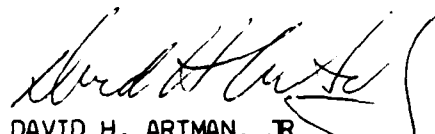
This report has been authored by a contractor of the United States Government. Accordingly, the United States Government retains a nonexclusive, royalty-free license to publish or reproduce the material contained herein, or allow others to do so, for the United States Government purposes.

This report has been reviewed by the Public Affairs Office and is releasable to the National Technical Information Services (NTIS). At NTIS, it will be available to the general public, including foreign nations.

If your address has changed, if you wish to be removed from our mailing list, or if your organization no longer employs the addressee, please notify AFWL/NTES, Kirtland AFB, NM 87117-6008 to help us maintain a current mailing list.

This technical report has been reviewed and is approved for publication.

  
TIMOTHY J. ROSS  
Project Officer

  
DAVID H. ARTMAN, JR  
Major, USAF  
Chief, Applications Branch

FOR THE COMMANDER

  
MAYNARD A. PLAMONDON  
Chief, Civil Engrg Rsch Div

DO NOT RETURN COPIES OF THIS REPORT UNLESS CONTRACTUAL OBLIGATIONS OR NOTICE ON A SPECIFIC DOCUMENT REQUIRES THAT IT BE RETURNED.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

## REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION Unclassified			1b. RESTRICTIVE MARKINGS <b>A174503</b>		
2a. SECURITY CLASSIFICATION AUTHORITY			3. DISTRIBUTION / AVAILABILITY OF REPORT Approved for public release; distribution unlimited.		
2b. DECLASSIFICATION / DOWNGRADING SCHEDULE					
4. PERFORMING ORGANIZATION REPORT NUMBER(S) 5956			5. MONITORING ORGANIZATION REPORT NUMBER(S) AFWL-TR-86-22		
6a. NAME OF PERFORMING ORGANIZATION Applied Research Associates, Inc		6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Weapons Laboratory		
6c. ADDRESS (City, State, and ZIP Code) 4300 San Mateo Blvd, NE, Suite A220 Albuquerque, New Mexico 87110			7b. ADDRESS (City, State, and ZIP Code) Kirtland Air Force Base, NM 87117-6008		
8a. NAME OF FUNDING / SPONSORING ORGANIZATION		8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER F29601-85-C-0029		
8c. ADDRESS (City, State, and ZIP Code)			10. SOURCE OF FUNDING NUMBERS		
			PROGRAM ELEMENT NO 62601F	PROJECT NO. 8809	TASK NO 13
					WORK UNIT ACCESSION NO 95
11. TITLE (Include Security Classification) HIGH FREQUENCY ANALYSIS OF CIRCULAR ARCHES					
12. PERSONAL AUTHOR(S) Merkle, Douglas Hall; Merkle, Laurence Douglas					
13a. TYPE OF REPORT Final		13b. TIME COVERED FROM 1 Feb 85 to 4 Feb 86		14. DATE OF REPORT (Year, Month, Day) 1986, September	
15. PAGE COUNT 134					
16. SUPPLEMENTARY NOTATION					
17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB-GROUP			
13	13		Arches Rotary Inertia		
			Dynamic Analysis Modal Analysis		
			Shear Deformation Frequency Equation		
19. ABSTRACT (Continue on reverse if necessary and identify by block number) This report presents the theory of the dynamic response of a linearly elastic, circular arch, including the effects of both shear deformation and rotary inertia. The arch central angle is arbitrary, and the end restraints are linearly elastic. The theory is extended to include linear viscoelastic behavior, in which both arch and end restraints are governed by the same single linear rate sensitivity parameter.					
20. DISTRIBUTION / AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS			21. ABSTRACT SECURITY CLASSIFICATION Unclassified		
22a. NAME OF RESPONSIBLE INDIVIDUAL Dr Timothy J. Ross			22b. TELEPHONE (Include Area Code) (505) 844-9087		22c. OFFICE SYMBOL NTES

DD FORM 1473, 84 MAR

83 APR edition may be used until exhausted.

All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

# SUMMARY

This report presents the theory of the dynamic response of a linearly elastic, circular arch, including the effects of both shear deformation and rotary inertia. The arch central angle is arbitrary, and the end restraints are linearly elastic. The theory is extended to include linear viscoelastic behavior, in which both arch and end restraints are governed by the same single linear rate sensitivity parameter.

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
Availability Codes	
/or	
Dist	
A-1	



## TABLE OF CONTENTS

<u>Section</u>	<u>Title</u>	<u>Page</u>
1	Introduction	1
2	Discussion	2
3	Conclusions	6
4	References	6
5	Bibliography	8
6	List of Symbols	17

### APPENDICES

A	Governing Equations for the Dynamic Response of a Circular Arch, Including Both Shear and Rotary Inertia	23
B	Matrix Form of the Governing Equations	38
C	Uncoupling the Equations of Motion	43
D	Expansion of the Characteristic Determinant	47
E	General Solution of the Characteristic Differential Equation	56
F	Characteristic Mode Functions	62
G	Symmetry	93
H	Mode Shape Determination	95
I	Orthogonality Relations	99
J	Transient Analysis	108
K	Viscoelastic Analysis	110
L	Numerical Analysis	113



# LIST OF FIGURES

<u>Figure</u>	<u>Title</u>	<u>Page</u>
1	Definition of Terms of Controlling the Dynamic Response of a Circular Arch	3
A1	Differential Element of a Circular Arch	24
A2	Normal Cross Section	26
A3	Differential Geometry in the Undeformed Arch	27
A4	Shear Stress Directions for a Deformed Arch Element when the Angle $Q_1^*Q_2^*$ is Acute	32
E	Argand Diagram Showing $A^3$ and $B^3$ when $R < 0$	60

## 1.0 INTRODUCTION

### 1.1 Purpose

This report presents the theory of the dynamic response of a linearly elastic, circular arch, including the effects of both shear deformation and rotary inertia. The arch central angle is arbitrary, and the end restraints are linearly elastic. The theory is extended to include linear viscoelastic behavior, in which both arch and end restraints are governed by the same single linear rate sensitivity parameter.

### 1.2 Motivation

The work described herein is an analytical extension of results presented in Reference 1. The approach was motivated by Dr. Timothy Ross' success in explaining direct shear failure of dynamically loaded reinforced concrete box roof slabs using a Timoshenko beam model and an appropriate reinforced concrete direct shear failure criterion (Ref. 2).

### 1.3 Application

When the theoretical results presented herein have been evaluated numerically for prescribed arch central angle, end restraint, and loading, they will apply to the KACHINA arches tested at the Air Force Weapons Laboratory (Refs. 3-5). Admittedly reinforced concrete is not always linearly elastic, so application of the theory to the KACHINA arches will require judgment. Nevertheless, the convenience of a closed form solution afforded by elastic theory is too attractive not to exploit for the perspective it gives on overall dynamic behavior and the influence of structural and loading parameters.

### 1.4 Point of Departure

The point of departure for the dynamic analysis is the set of equations for the static behavior of a circular planar member presented by Connor in Chapter 14 of Reference 6. Other related work is presented in References 7-11.

Details of the dynamic analysis are contained in Appendices A-L. The discussion in Section 2 below summarizes each appendix in order. Because of the detailed treatment and the many equations, a few letter symbols have been used differently in separate appendices. However, symbols are defined where first used in each appendix, so there should be no confusion. See Appendix M.

## 2.0 DISCUSSION

Figure 1 shows a portion of a circular arch, and defines the terms which control its dynamic response. The arch centerline radius is  $R$ , and the central angle defining a particular plane normal cross section,  $PQ$ , in the undeformed arch is  $\theta$ . The width and depth of the constant rectangular arch cross section are  $b$  and  $d$ . Deformation of the arch causes the centroid of a plane cross section  $PQ$  to displace tangentially by an amount  $u_1$  and radially inward by an amount  $u_2$ , and the cross section to rotate through an angle  $\psi$ ; but the section is assumed to remain plane, so that  $P^*Q^*$  is a straight line. The distributed tangential, radial, and moment loads acting at point  $O$  in the undeformed arch are  $b_1$ ,  $b_2$ , and  $m$ . Notice that the angle  $\psi$  is an independent displacement parameter, so that  $P^*Q^*$  is generally not normal to the deformed centroidal axis. The angular amount by which section  $P^*Q^*$  deviates from being normal to the deformed centroidal axis is the arch shear deformation at point  $O$ .

Appendix A begins by formulating the equations of motion for a differential arch element, using the internal stress resultants (Equations A14-A16). Next, the form of the displacement field is defined by assuming that plane cross sections normal to the undeformed arch centroidal axis remain plane during deformation. This is equivalent to expanding the displacements in a Taylor series in two variables about the centroid (arc length and inward radial position) and retaining only the constant and linear terms. Having assumed the form of the displacement field (Equation A23), the extensional and shear strains are obtained as functions of the displacement field parameters and their derivatives with respect to arc length (Equations A31-A33). Three

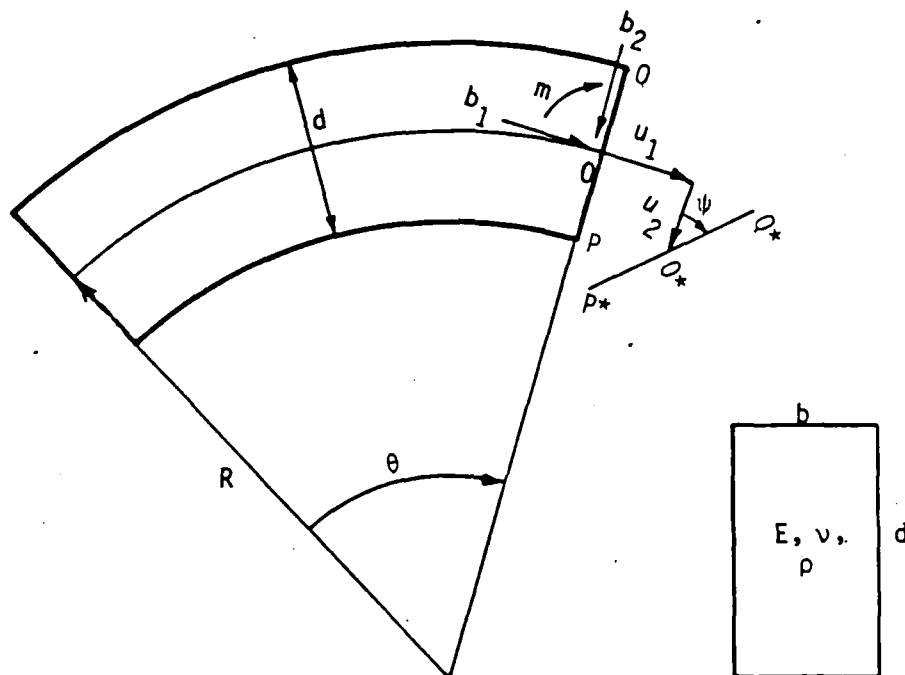


Figure 1. Definition of Terms Controlling the Dynamic Response of a Circular Arch.

strain parameters are defined at the centroid (Equations A34-A36), and then the strain components anywhere in the arch are expressed as functions of the strain parameters (Equations A37-A39). Normal and shear stresses are obtained as functions of the strain parameters, using elastic stress-strain equations (Equations A40 and A41), and then integrated over the cross section to obtain the internal stress resultants as functions of the strain parameters (Equations A42-A44). The stress resultant equations of motion, stress resultant-strain component equations, and strain component-displacement equations are summarized in matrix form in Equations A61, A63, and A64. Use of matrix notation greatly simplifies the entire analysis. It is compact, helps avoid algebraic errors, and even suggests analytical approaches not obvious when equations are written out in detail.

The equations of motion in terms of displacement components and their derivatives are obtained in matrix form in Appendix B, by substituting Equation A64 into Equation A63 and that result into Equation A61. The final result is Equation B17.

Appendix C considers the homogeneous (free vibration) form of the equations of motion (C2), and uses the classic separation of variables approach to define a vibration mode (Equation C3). Free vibration in a single mode is shown to be harmonic (Equation C15), and the mode shapes are shown to obey a coupled set of three, linear, ordinary, second order differential equations (C18). The three, coupled, mode shape differential equations are uncoupled by pre-multiplying them by the adjoint of the coefficient operator matrix (Ref. 12), but at the cost of having the resulting three separate identical linear ordinary differential equations be of order six instead of two (Equations C21-C23).

Equation C20 defines the sixth order, linear, ordinary differential operator,  $\Delta$ , which appears in each of Equations C21-C23. These are the spatial differential equations which the mode shapes must satisfy. The operator,  $\Delta$ , is defined as a determinant, which when expanded yields a sixth order polynomial in the frequency parameter,  $\sigma$ , (Equation D17) or a sixth order polynomial in the spatial differential operator,  $D$ , (Equation D18). Equation D18 is the more fundamental form, and shows the frequency dependence of its coefficients.

The last half of Appendix D discusses the way in which the coefficients in Equation D18 vary with frequency.

The operator,  $\Delta$ , defined by Equation C20 is a sixth order, linear, ordinary differential operator, but the odd order derivatives are missing. Therefore, assuming the solution of the homogeneous equation  $\Delta X = 0$  (Equation C21) to be of the form  $X = Ke^{\lambda\theta}$  yields a cubic characteristic equation in  $\lambda^2$  (E6). Whether the parameter  $\lambda$  is real, imaginary, or complex depends on the coefficients of the cubic equation. Appendix E discusses the algebraic solution of a cubic equation, as the basis for a detailed examination of the nature of the characteristic roots and their associated mode shape functions, which appears in Appendix F.

Appendix F examines the fifteen possible characteristic root combinations and their associated mode functions. The mode functions are a closed set of six linearly independent functions, in the sense that differentiation of any one yields a linear combination of the other five.

Appendix G establishes the derivative properties of symmetric and antisymmetric functions needed to define arch centerline boundary conditions for symmetric and antisymmetric modes.

Appendix H explains in detail how the natural frequencies and associated mode shapes are calculated for given arch boundary conditions. The feature which distinguishes this modal analysis from many others is the fact that the precise form of the frequency equation is frequency dependent.

Appendix I presents the arch modal orthogonality relations. The approach is that for a Sturm-Liouville problem, supplemented by Rayleigh's ingenious application of L'Hospital's Rule to find the integral of the inner product of a mode with itself over the arch length (Refs. 13 and 14).

Once the free vibration mode orthogonality relations have been established, the formal solution of a transient forced vibration problem becomes straightforward. Both the arch displacements and distributed loads are assumed to be expressible as modal series expansions, and the equations of motion are then scanned with a particular mode shape. The result is a single degree of freedom differential equation for the associated modal amplitude (J6), the solution of which is a Duhamel convolution integral (Equation J8).

Appendix K extends the above elastic arch analysis to the case of viscoelastic behavior. When both arch and boundary restraints are governed by the same single rate sensitivity parameter, the elastic modal analysis still applies.

Appendix L addresses the most difficult computational phase of arch modal analysis, finding the roots of the frequency equation (H7). The frequency equation is a complicated transcendental equation, usually involving both trigonometric and hyperbolic terms. The modal frequency parameter,  $\sigma$ , not only appears implicitly in several places, but also controls the precise form of the frequency equation. Considerable care is needed in calculating the higher modal frequencies, because some of them are apt to be closely spaced.

### 3.0 CONCLUSIONS

The distinguishing feature of the closed form, dynamic arch analysis presented in this report is its extensive use of matrix notation. The analysis includes the effects of both shear deformation and rotary inertia on the transient response of both an elastic and a viscoelastic circular arch, having an arbitrary central angle and corresponding elastic or viscoelastic end restraints. The resulting equations are presented at a level of detail sufficient for direct computer programming, and a computer program (a modification of ZEROIN) is presented for finding the modal frequencies.

### 4.0 REFERENCES

1. Auld, H. E., W. C. Dass, and D. H. Merkle, Development Of Improved SDOF Analysis Procedures For Buried Arch Structures, AFWL-TR-83-39, Air Force Weapons Laboratory, Kirtland Air Force Base, NM, (Jun 1983).
2. Ross, T. J., Direct Shear Failure in Reinforced Concrete Beams Under Impulsive Loading, AFWL-TR-83-84, Air Force Weapons Laboratory, Kirtland Air Force Base, NM, (Sep 1983).
3. Parsons, R., and E. Rinehart, KACHINA Test Series: BUTTERFLY MAIDEN Quick Look Report, two volumes, AFWL-TR-82-132, Air Force Weapons Laboratory, Kirtland Air Force Base, NM, (Jun 1983).
4. Betz, J. F., J. Edwards, J. Smith, J. Verner, and G. Walhood, KACHINA Test Series: EAGLE DANCER, three volumes, AFWL-TR-83-35, Air Force Weapons Laboratory, Kirtland Air Force Base, NM, (Jul 1983).

5. Smith, J. L., J. F. Betz, C. M. Hazen, K. Havens, and T. Steiner, KACHINA Test Series: DYNAMIC ARCH TEST-3 Pretest Report, AFWL-TR-83-56, Air Force Weapons Laboratory, Kirtland Air Force Base, NM, (Sep 1983).
6. Connor, J. J., Analysis of Structural Member Systems, Ronald Press, (1976).
7. Wung, S. J., Vibration of Hinged Circular Arches, MS thesis, Rice University, (May 1967).
8. Pereira, C. A. L., Free Vibration of Circular Arches, MS thesis, Rice University, (1968).
9. Veletsos, A. S., W. J. Austin, C. A. L. Pereira, and S. J. Wung, Free In-Plane Vibration of Circular Arches, ASCE PROC., Vol. 98, No. EM2, pp. 311-329, (Apr 1972).
10. Austin, W. J. and A. S. Veletsos, Free Vibration of Arches Flexible in Shear, ASCE PROC., Vol. 99, No. EM4, pp. 735-753, (Aug 1973).
11. Henrych, J., The Dynamics of Arches and Frames, Elsevier, (1981).
12. Latta, G. and J. McGregor, The Fundamental Solution Matrix for Systems of Linear Partial Differential Equations, Stanford University report to the Wright Air Development Center, WADC-TR-58-416, AD 155824, (Aug 1958).
13. Rayleigh, J. W. S., The Theory of Sound, Vol. 1, Dover, (1945).
14. Prescott, J., Applied Elasticity, Dover, (1961).



## 5.0 BIBLIOGRAPHY

Albright, G.H., E.J. Beck, J.C. Le Doux, and R.A. Mitchell, Evaluation of Buried Corrugated Steel Arch Structures, Operation PLUMBBOB Project 3.3, WT-1422, Hq. DASA, (Feb 1961).

Allgood, J.R. "Blast Loading of Small Buried Arches," ASCE Proceedings, Vol. 90, No. ST5, pp. 39-61, (Oct 1964); also US Naval Civil Engineering Laboratory report TR-R-216, (Apr 1963).

Allgood, J.R. and W.O. Carter, Predicting Blast-Induced Body Motions of a Buried Structure With Footings, US Naval Civil Engineering Laboratory report TR-R-539, (Aug 1967).

Allgood, J.R., and D. DaDeppo, Body Motions of a Buried Arch Subjected to Blast Loading, US Naval Civil Engineering Laboratory Report TN-486, (Aug 1963).

Allgood, J.R., and R.H. Seabold, "Body Rotations of Buried Arches, Project 3.4," in Proceedings of Symposium on Operation SNOWBALL, Vol. 1, US Naval Civil Engineering Laboratory report to the Defense Atomic Support Agency, DASA 1642-1, (Aug 1965).

Allgood, J.R., and R.H. Seabold, Shallow Buried Model Arches Subjected to a Traveling Wave Load, US Naval Civil Engineering Laboratory report to the Defense Atomic Support Agency, DASA-13.161, NCEL TR R-375, (Oct 1965).

Allgood, J.R., C.R. White, R.F. Swalley, and H.L. Gill, Blast Loading of Small Buried Arches, US Naval Civil Engineering Laboratory report TR-216, (Mar 1963).

Anderson, R.H., R.T. Haelsig, and M.D. Reifel, Structural Behavior of Ring Sections Under Nonuniform External Pressure, AFWL-TR-65-145, Air Force Weapons Laboratory, Kirtland Air Force Base, (Mar 1966).

Archer, R.R., "Small Vibrations of Thin Incomplete Circular Rings," International Journal of Mechanical Sciences, Vol. I, No. 1, pp. 45-56, (1960).

Austin, W.J., "In-Plane Bending and Buckling of Arches," ASCE Proceedings, Vol. 97, No. ST5, pp. 1575-1592, (May 1971); disc. Vol. 98, No. ST1, pp. 373-379, (Jan 1972); closure Vol. 98, No. ST7, pp. 1670-1672, (Jul 1972).

Austin, W.J., and T.J. Ross, "Elastic Buckling of Arches Under Symmetrical Loading," ASCE Proceedings, Vol. 102, No. ST5, pp. 1085-1095, (May 1976).

Austin, W.J., T.J. Ross, A.S. Tawfik, and R.D. Volz, "Numerical Bending Analysis of Arches," ASCE Proceedings, Vol. 108, No. ST4, pp. 849-868, (Apr 1982).

Austin, W.J. and A.S. Veletsos, "Free Vibration of Arches Flexible in Shear," ASCE Proceedings, Vol. 99, No. EM4, pp. 735-753, (Aug 1973).

Ball, R.E., "Dynamic Analysis of Rings by Finite Differences," ASCE Proceedings, Vol. 93, No. EM1, pp. 1-10, (Feb 1967).

Balsara, J.P., Similitude Study of Flexible Buried Arches Subjected to Blast Loads, US Army Engineer Waterways Experiment Station report TR I-807, (Jan 1968).

Bellow, D.G. and A. Semeniuk, "Symmetrical and Unsymmetrical Forced Excitation of Thin Circular Arches," International Journal of Mechanical Sciences, Vol. 14, No. 3, pp. 185-195, (Mar 1972).

Biggs, J.M., Introduction to Structural Dynamics, McGraw-Hill, (1964).

Blevins, R.D., Formulas for Natural Frequency and Mode Shape, Van Nostrand Reinhold, (1979).

Breckenridge, R.A., J.R. Allgood, and R.M. Webb, "Shallow Underground Arches as Shelters," Shock Vibration and Associated Environments, Bull. No. 28, Part III, pp. 252-268, (Sep 1960).

Buckens, F. "Influence of the Relative Radial Thickness of a Ring on its Natural Frequencies," Journal ASA, Vol. 22, No. 4, pp. 437-443, (Jul 1950).

Bultmann, E.H., T.G. Morrison, and M.R. Johnson, Full-Scale Field Tests of Dome and Arch Structures, Operation PLUMBBOB, Project 3.6, Mechanics Research Division, American Machine and Foundry Company and Air Force Special Weapons Center, weapon test report WT-1425, (Aug 1960).

Calhoun, P.R., and D.A. DaDeppo, "Nonlinear Finite Element Analysis of Clamped Arches," ASCE JSE, Vol. 109, No. 3, pp. 599-612, (Mar 1983).

Connor, J.J., Analysis of Structural Member Systems, Ronald Press, (1976).

Costes, N.C., "Factors Affecting Vertical Loads on Underground Ducts Due to Arching," HRB Bulletin No. 125.

Crawford, R.E., C.J. Higgins, and E.H. Bultmann, The Air Force Manual For Design and Analysis of Hardened Structures, AFWL-TR-74-102, Air Force Weapons Laboratory, Kirtland Air Force Base, NM, (Oct 1974).

Crespo Da Silva, M.R.M., "Vibrations of Shallow Arches, Including the Effect of Geometric Non-Linearities," Journal of Sound and Vibration, Vol. 84, pp. 161-172, (Sep 1982).

Crowson, R.D., ESSEX-DIAMOND ORE Research Program: Aircraft Shelter Response, Project ESSEX-V, US Army Engineer Waterways Experiment Station Technical Report N-77-3, (Dec 1977).

Crowson, R.D., Mechanical Impedance Tests of Prototype and Model Concrete Arch Aircraft Shelters, US Army Engineer Waterways Experiment Station report to the Defense Nuclear Agency, WES Technical Report N-75-6, (Nov 1975).

Crowson, R.D., Role of Forced Vibration in Structural Dynamics, MS thesis, Mississippi State University, Starkville, Mississippi, (Aug 1978).

DaDeppo, D.A., T.E. Sherlock, and R.D. Tazelaar, Protective Arch Structures, University of Michigan Research Institute report to the Air Force Special Weapons Center, AFSWC TR 58-28, (Mar 1958).

Dawkins, W.P., "Analysis of Tunnel Liner-Packing Systems," ASCE Proceedings, Vol. 95, No. EM3, pp. 679-693, (Jun 1969); disc. Vol. 95, No. EM6, pp. 1447-1451, (Dec 1969); closure Vol. 96, No. EM4, pp. 499-500, (Aug 1970).

Den Hartog, J.P., "The Lowest Natural Frequency of Circular Arcs," Philosophical Magazine, 7th Series, Vol. V, No. 28, p. 400-408, (Feb 1928).

Denton, D.R., and W.J. Flathau, "Model Study of Dynamically Loaded Arch Structures," ASCE Proceedings, Vol. 92, No. EM3, pp. 17-32, (June 1966).

Eppink, R.T., and A.S. Veletsos, Analysis and Design of Domes, Arches and Shells: Vol. II--Analysis of Circular Arches, University of Illinois report to the Air Force Special Weapons Center, AFSWC TR 59-9, (Jul 1959); U of I SRS No. 186.

Eppink, R.T., and A.S. Veletsos, Analysis and Design of Domes, Arches and Shells: Vol. II--Dynamic Response of Circular Arches Under Uniform All-Around Pressure Pulse, University of Illinois report to the Air Force Special Weapons Center, AFSWC TR 60-16, (Oct 1960).

Eppink, R.T., and A.S. Veletsos, "Dynamic Analysis of Circular Elastic Arches," Proc. 2nd Conf. on Electronic Computation, ASCE, pp. 477-502, (Sep 1960).

Eppink, R.T., and A.S. Veletsos, Response of Arches Under Dynamic Loads, University of Illinois report to the Air Force Special Weapons Center, AFSWC-TR-60-53, (Dec 1960).

Federhofer, K., Dynamik des Bogenträgers und Kreisringes, Springer-Verlag, Wien, (1950).

Federhofer, K., "Über den Einfluss der Achsendehnung der Rotationstragheit und der Schubkraft auf die Frequenzen der Biegungsschwingungen eines Kreisringes," S.B. Akad. Wiss. Wien, Series IIa, Vol. 144, pp. 307-315, (1935).

Felgar, R.P., Jr., Formulas for Integrals Containing Characteristic Functions of a Vibrating Beam, Circular No. 14, University of Texas Bureau of Engineering Research, Austin, Texas, (1950).

Flathau, W.J., R.A. Breckenridge, and C.K. Wiehele, Blast Loading and Response of Underground Concrete--Arch Protective Structures, Operation PLUMBBOB, Project 3.1, US Army Engineer Waterways Experiment Station and US Naval Civil Engineering Laboratory, WT 1420, (Jun 1959).

Flathau, W.J., R.A. Sager, and F.A. Luzi, Design and Analysis of Underground Reinforced Concrete Arches, US Army Engineer Waterways Experiment Station report TR 2-590, (Jan 1962).

Flügge, W., Stresses in Shells, Springer-Verlag, (1960).

Frank, R.A., Transfer Function and Fragility Algorithm for Shallow-Buried R/C Arch, Applied Research Associates draft interim report to the Air Force Weapons Laboratory for Contract F29601-83-C-0030, (Jan 1985).

Fulton, R.E., and F.W. Barton, "Dynamic Buckling of Shallow Arches," ASCE Proceedings, Vol. 97, No. EM3, pp. 865-877, (Jun 1971); disc. Vol. 98, No. EM2, pp. 478-481, (Apr 1972).

Gallagher, E.V., Air Blast Loading on Arches and Domes, Armour Research Foundation Final Test Report No. 13 to the Air Force Special Weapons Center, AFSWC-TN-58-26, (Sep 1958).

Gill, H.L., and J.R. Allgood, Static Loading of Small Buried Arches, US Naval Civil Engineering Laboratory report R-278, (Jan 1964).

Goldberg, J.E., J.L. Bogdanoff, and W.D. Glanz, "General Computer Analysis of Beams," ASCE Proceedings, Vol. 90, No. EM3, pp. 135-146, (Jun 1964).

Gregory, W.E., and R.H. Plaut, "Dynamic Stability Boundaries for Shallow Arches," ASCE Proceedings, Vol. 108, No. EM6, pp. 1036-1050, (Dec 1982).

Grubaugh, R.E., T.G. Morrison, R.S. Kocke, G.L. Neidhardt, and W. Tuggle, Full Scale Field Tests of Dome and Arch Structures, Operation PLUMBBOB, ITR 1425, (Oct 1957).

Gupchup, V.N., Nonlinear Response of Two-Hinged Circular Reinforced Concrete Arches to Static and Dynamic Loads, DSc thesis, M.I.T., (Jan 1963).

Gupchup, V.N., and J.M. Biggs, "Dynamic Nonlinear Response of Reinforced Concrete Arches," ASCE Proceedings, Vol. 89, No. ST4, pp. 225-248, (Aug 1963); disc. Vol. 90, No. ST1, p. 257, (Feb 1964); closure Vol. 90, No. ST4, pp. 261-265, (Aug 1964).

Hansen, R.J., and H.D. Smith, Response of Buried Arch and Dome Models, Operation SUNBEAM, Shot SMALL BOY, Project 3.1, Massachusetts Institute of Technology report to the Defense Atomic Support Agency, POR 2222, (Aug 1962).

Harris, J.F., and A.R. Robinson, Numerical Methods for the Analysis of Buckling and Post Buckling Behavior of Arch Structures, University of Illinois SRS No. 364, (1967).

Harrison, H.B., "In-Plane Stability of Parabolic Arches," ASCE Proceedings, Vol. 108, No. ST1, pp. 195-205, (Jan 1982).

Heger, F.J., "Structural Behavior of Circular Reinforced Concrete Pipe--Development of Theory," Journal of the American Concrete Institute, pp. 1567-1614, (Nov 1963).

Hencky, H., "Stabilitätsprobleme der Elastizitätstheorie," Zeitschrift für Angewandte Mathematik und Mechanik, Bd. 2, Heft 4, pp. 292-299, (Aug 1922).

Henrych, J., The Dynamics of Arches and Frames, Elsevier, (1981).

Hoppe, R., "Vibrationen eines Ringes in seiner Ebene," Journal für Mathematik (Crelle), Bd. 73, pp. 158-170, (1871).

Hsu, C.S., C.T. Kuo, and S.S. Lee, "On the Final States of Shallow Arches on Elastic Foundations Subjected to Dynamical Loads," ASME JAM, Vol. 35, pp. 713-723, (Dec 1968).

Huang, T., S. Iyengar, and R.L. Jennings, Studies of Response of Arches and Domes Under Dynamic Loads, University of Illinois report to the Air Force Special Weapons Center, AFSWC-TR-61-90, (Oct 1961).

Irie, T., G. Yamada, and K. Tanaka, "Natural Frequencies of Out-of-Plane Vibration of Arches," ASME JAM, Vol. 49, pp. 910-913, (Dec 1982).

Isenberg, J., H.S. Levine, and S.H. Pang, Numerical Simulation of Forced Vibration Tests on a Buried Arch, Weidlinger Associates report to the Defense Nuclear Agency, DNA 4281F, (Mar 1977).

Kennedy, T.E., "Comparison of Simulation and Field Tests of a Buried Concrete Arch Structure," Proceedings of the Conference on Military Application of Blast Simulators, (Jul 1967); also US Army Engineer Waterways Experiment Station Miscellaneous Paper I-963, (Jan 1968).

Kennedy, T.E., Dynamic Response of a Model Buried Field Shelter, Project LN314, Operation PRAIRIE FLAT, US Army Engineer Waterways Experiment Station report TR N-70-6, (Mar 1970).

Kennedy, T.E., Dynamic Tests of a Model Flexible Arch Type Protective Shelter, US Army Engineer Waterways Experiment Station Miscellaneous Paper N-71-3, (Apr 1971).

Kennedy, T.E., "Project LN314A Evaluation of Protective Military Shelters," Event DIAL PACK Symposium Report, Vol. II, US Army Engineer Waterways Experiment Station report, (Mar 1971).

Kennedy, T.E., The Dynamic Response of a Simulated Buried Arch to Blast Loading, US Army Engineer Waterways Experiment Station report TR N-71-9, (Jul 1971).

Kennedy, T.E., and Ballard, J.T., Dynamic Tests of a Model Flexible Arch Type Protective Shelter, US Army Engineer Waterways Experiment Station report TR I-768, (Apr 1967).

Kirkhope, J., "Simple Frequency Expression for the In-Plane Vibration of Thick Circular Rings," Journal ASA, Vol. 59, No.1, pp. 86-88, (Jan 1976).

Laing, E.B., and E. Cohen, "Design of Below Ground Arch and Dome Type Structures Exposed to Nuclear Blast," Shock Vibration and Associated Environments, Bull. 29, Part III, pp. 188-223, (Jul 1961).

Lamb, H., "On the Flexure and Vibrations of a Curved Bar," Proceedings London Math. Soc., Vol. 19, No. 328, pp. 365-376, (May 1888).

LeDoux, J.C., and P.J. Rush, Response of Earth-Confined Flexible Arch Structures in High Overpressure Regions, Project 3.2, Operation HARDHAT, Draft of WT-1626, US Naval Civil Engineering Laboratory, (no date).

Leontovich, V., Frames and Arches, McGraw-Hill, (1959).

Lipner, N., W.A. Millavec, R.E. Sawyer, R.C. Shaw, J. Karagozian, and W.T. Tsai, Fragility Analysis of Hardened Structures, AFWL-TR-81-231, Air Force Weapons Laboratory, Kirtland Air Force Base, NM, (Aug 1982).

Lock, M.H., "Effect of Rise Time on Critical Dynamic Load," AIAA Journal, Vol. 6, pp. 162-164.

Love, A.E.H., A Treatise on the Mathematical Theory of Elasticity, Dover, pp. 451-454, (1944).

Mayer, R., "Über Elastizität des Geschlossenen und Offenen Kreisbogens," Zeitschrift für Mathematik und Physik, Bd. 61, pp. 246-320, (1913).

McClellan, R.E., PRE-DICE THROW II-I, Structures Experiments, POR 6097, Air Force Space and Missile Systems Organization, Norton Air Force Base, CA, (Jan 1976).

McGrath, R.K., Dynamic Response of Concrete Arch Bunkers, Event DIAL PACK, Project LN314A, US Army Engineer Waterways Experiment Station report TR N-71-8, (Jul 1971).

McGrath, R.K., Dynamic Response of Concrete Arch Bunkers, Laboratory Tests, US Army Engineer Waterways Experiment Station Miscellaneous Paper, (Apr 1971).

Mebane, P.M., and J.A. Stricklin, "Implicit Rigid Body Motion in Curved Finite Elements," AIAA Journal, Vol. 9, No. 2, pp. 344-345, (1971).

Meyer, G.D., and W.J. Flathau, Static and Dynamic Laboratory Tests of Unreinforced Concrete Fixed-End Arches Buried in Dry Sand, US Army Engineer Waterways Experiment Station report TR-I-758, (Feb 1967).

Morley, L.S.D., "The Flexible Vibrations of a Cut Thin Ring," Quart. Journal Mechanical and Applied Math., Vol. XI, Part 4., pp. 491-497, (1958).

Morrison, T.G., Protective Construction Final Report Part III of III--Mathematical Analyses, American Machine and Foundry Company Mechanics Research Division report to the Air Force Special Weapons Center, AFSWC TR-59-2, Part III, (Dec 1958).

Munn, J.F., G.L. Carre, and T.E. Kennedy, Failure of Footing-Supported Buried Steel Arches Loaded Staticaly, US Army Engineer Waterways Experiment Station Miscellaneous Paper N-70-2, (Mar 1970).

Nash, P.T., and J.H. Smith, "Dynamic Response of a Soil Covered Concrete Arch to Impact and Blast Loadings," S & V BULL., Part 4, pp. 67-74, (Sep 1977).

Nelson, C.R., The Dynamic Response of Reinforced Concrete Circular Arches, Massachusetts Institute of Technology report to the US Naval Civil Engineering Laboratory, R64-37, (Aug 1964).

Nelson, F.C., "Frequency Cross-Over of Simply Supported Circular Ring Segments," Journal Franklin Institute, Vol 278, No. 1, pp. 20-27, (Jul 1964).

Nelson, F.C., "In-Plane Vibration of a Simply Supported Circular Ring Segment," International Journal of Mechanical Sciences, Vol. 4, No. 5., pp. 517-527, (Nov-Dec 1962).

Newmark, N.M., Vulnerability of Arches, Preliminary Notes, paper for Physical Vulnerability Division, USAF, (rev. 21 May 1956).

Newmark, N.M., J.W. Briscoe, and J.L. Merritt, Analysis and Design of Flexible Underground Structures, Vol. 1, University of Illinois report to the US Army Engineer Waterways Experiment Station, (May 1960).

Newmark, N.M., and Haltiwanger, J.D., Air Force Design Manual: Principles and Practices For Design of Hardened Structures, University of Illinois report to the Air Force Special Weapons Center, AFSWC-TDR-62-138, (Dec 1962).

Oran, C., "General Imperfection Analysis in Shallow Arches," ASCE Proceedings, Vol. 106, No. EM6, pp. 1175-1193, (Dec 1980).

Palacios, N., and T.E. Kennedy, The Dynamic Response of Buried Concrete Arches, Operation SNOWBALL, Project 3.2, US Army Engineer Waterways Experiment Station report TR-I-797, (Sep 1967).

Pang, S.H. and J. Isenberg, ESSEX-DIAMOND ORE Research Program: Structure-Medium Interaction and Structural Response Calculations for the ESSEX Phase 3 Event, Weidlinger Associates report to the US Army Engineer Waterways Experiment Station, TR N-78-3, (May 1978).

Pereira, C.A.L., Free Vibration of Circular Arches, MS Thesis, Rice University, (1968).

Philipson, L.L., "On the Role of Extension in the Flexural Vibration of Rings," ASME JAM, Vol. 23, No. 3, pp. 364-366, (Sep 1956).

Philipchuk, V.N., "Concerning the Essentially Nonlinear Dynamics of Arches and Rings," PMM, Vol. 46, pp. 461-466, (May-Jun 1982), (in Russian).

Prager, W., Introduction to Mechanics of Continua, Ginn and Company, (1961).

Prescott, J., Applied Elasticity, Dover, pp. 307-310, (1961).

Raithel A., and C. Franciosi, "Dynamic Analysis of Arches Using Lagrangian Approach," ASCE JSE, Vol. 110, No. 4, pp. 847-858, (Apr 1984).

Rao, S.S., and V. Sundararajan, "In-Plane Flexural Vibrations of Circular Rings," ASME JAM, Vol. 36, No. 3, pp. 620-625, (Sep 1969).

Robbins, D.T., and Williamson, R.A., Post Shot Analysis for Project 3.1: Operation PLUMBBOB, Holmes and Narver report to the US Army Engineer Waterways Experiment Station, HN-80-1020C, (Apr 1958).

Ross, T.J., Direct Shear Failure in Reinforced Concrete Beams Under Impulsive Loading, AFWL-TR-83-84, Air Force Weapons Laboratory, Kirtland Air Force Base, NM, (Sep 1983).

Ross, T.J., Numerical Large Deflection Bending and Buckling Analysis of Arches, MS thesis, Rice University, (1973).

Safford, F., Response Prediction and Induced Motion Simulation for Measured and Scaled System Functions, notes for short course at Pennsylvania State University, (Aug 1977).

Sager, R.A., Concrete Arch Studies, Operation SNOWBALL, Project 3.2, US Army Engineer Waterways Experiment Station Miscellaneous Paper I-736, (Aug 1965).

Schmidt, R., and D.A. DaDeppo, "A Survey of Literature On Large Deflections of Nonshallow Arches--Bibliography of Finite Deflections of Straight and Curved Beams, Rings, and Shallow Arches," Industrial Mathematics, Journal of the Industrial Mathematics Society, Vol. 21, Part 2, pp. 91-114, (1971).

Stanton, R.G., Numerical Methods for Science and Engineering, Prentice-Hall pp. 151-154, (1961).

Sundararajan, V., and D.S. Kumani, "Dynamic Snap-Buckling of Shallow Arches Under Inclined Loads," AIAA Journal, Vol. 10, pp. 1090-1091, (Aug 1972).

Tener, R.K., Model Study of Buried Arch Subjected to Dynamic Loading, US Army Engineer Waterways Experiment Station report TR-660, (Nov 1964).

Timoshenko, S.P., "Kippsicherheit des Gekrümmten Stabes mit Kreisförmiger Mittellinie," Zeitschrift für Angewandte Mathematik und Mechanik, Bd. 3, Heft 5, pp. 358-362, (Oct 1923).

Timoshenko, S.P., and D.H. Young, "Vibration of a Circular Ring," Vibration Problems in Engineering, Third Edition, Van Nostrand, Para. 68, pp. 425-430, (1955).

Timoshenko, S.P., D.H. Young, and W. Weaver, Jr., "Vibrations of Circular Rings," Vibration Problems in Engineering, Fourth Edition, Wiley, Para. 5.22, pp. 476-481, (1974).

Veletsos, A.S., W.J. Austin, C.A.L. Pereira, and S.J. Wung, "Free In-Plane Vibration of Circular Arches," ASCE Proceedings, Vol. 98, No. EM2, pp. 311-329, (Apr 1972).

Volterra, E., and J. D. Morell, "A Note on the Lowest Natural Frequency of Elastic Arcs," ASME JAM, Vol. 27, No.4, pp. 744-746, (Dec 1960).

Walls, A.W., The Influence of Blast and Earth Pressure Loadings on the Dynamic Response of Flexible Two-Hinged Arches, PhD thesis, University of Illinois, (1960).

Waltring, F.W. "Schwingungszahlen und Schwingungsformen von Kreisbogenträgern," Ingenieur-Archiv, Vol. V, pp. 429-449, (1934).

Wang, T.M., and J.M. Lee, "Forced Vibrations of Continuous Circular Arch Frames," Journal of Sound and Vibration, Vol. 32, No. 2, pp. 159-173, (Jan 1974).

Wen, R.K., and J. Lange, "Curved Beam Element for Arch Buckling Analysis," ASCE Proceedings, Vol. 107, No. ST11, pp. 2053-2069, (Nov 1981).

Whipple, C.R., Numerical Studies of the Dynamic Response of Shallow Buried Arches to Blast Loading, University of Illinois draft report to the Air Force Special Weapons Center, contracts AF 29(601)-2591 and 4508, (1961).

Whipple, C.R., The Dynamic Response of Shallow Buried Arches Subjected to Blast Loading, PhD Thesis; University of Illinois report to the Air Force Special Weapons Center, contract AF29(601)-2591, (Jun 1961).

Wolf, J.A., Jr., "Natural Frequencies of Circular Arches," ASCE Proceedings, Vol. 97, No. ST9, pp. 2337-2350, (Sep 1971); disc. Vol. 98, No. ST5, pp. 1208-1210, (May 1972).

Wong, F.S., Transfer Function Analysis for Statistical Survivability/Vulnerability Assessment of Protective Structures, AFWL-TR-82-32, Air Force Weapons Laboratory, Kirtland Air Force Base, NM, (Oct 1982).

Wung, S.J., Vibration of Hinged Circular Arches, MS thesis, Rice University, (May 1967).



Young, D.H. and R.P. Felgar, Jr., "Tables of Characteristic Functions Representing Normal Modes of Vibration of a Beam," University of Texas Engineering Research Series, No. 44, Austin, Texas, (Jul 1949).

## 6.0 LIST OF SYMBOLS

Listed below are the symbols used in this report, and the page where each is introduced and defined. A semicolon indicates reuse of a symbol with a different definition.

<u>Symbol</u>	<u>Page</u>
A	23 ; 47 ; 57
$A_1$	58
$A_2$	58
$A_3$	58
$A_j$	59
$A^*$	102
B	48 ; 57
$B_1$	58 ; 75
$B_2$	58
$B_3$	59
$B_j$	59
$B_k$	61
$\underline{B}$	97
$B_{ij}$	100
C	48
$C_1$	35
$C_2$	35
$C_3$	36
$\underline{C}$	95
(C)	97
D	4, 36
E	3, 33 ; 48
$\underline{E}$	97
$E^*$	102
F	48
$\bar{F}$	23

<u>Symbol</u>	<u>Page</u>
$F_1$	23
$F_2$	23
$(F)$	38
$G$	33
$\underline{G}$	38
$\underline{H}$	40
$H_i$	108
$H_j$	108
$I$	25
$\underline{I}$	39
$I_1$	48
$I_2$	48
$I_3$	48
$I_1 (\underline{X})$	50
$K$	5, 50
$\underline{K}$	41
$K_0$	53
$K_1$	53
$K_2$	53
$K_3$	53
$[K]$	104
$L$	56; 93
$\underline{L}$	43
$M$	23
$\overline{M}$	23
$(N)$	103
$N_1$	103
$N_2$	103
$N_3$	103
$(N)_i$	104
$(N)_j$	104
$P$	57
$Q$	57
$R$	2; 57

<u>Symbol</u>	<u>Page</u>
S	64
$\underline{S}$	38
T	43
$T_i$	109
$T_j$	108
(U)	43
$[\bar{U}]$	44
$(U)_i$	99
$(U)_j$	99
$w_1$	63
$w_2$	63
$w_3$	63
$w_4$	63
$w_5$	63
$w_6$	63
(W)	95
$w_i$	96
X	5, 43
$X_i$	101
$X_j$	101
Y	43
$Y_i$	101
$Y_j$	101
Z	43
$Z_i$	101
$Z_j$	101
a	51
b	2; 51
$b_1$	2, 23
$b_2$	2, 23
$\bar{b}$	23
(b)	38
c	41; 54

<u>Symbol</u>	<u>Page</u>
d	2
$d^*$	33
e	5
f	104
$f(x)$	92
$f_E(x)$	92
$f_0(x)$	92
i	57
k	31
$k'$	37
m	2, 23
$\bar{m}$	23
(m)	43
n	52
p	43
$p_j$	106
r	36; 62
$\bar{r}$	25
$\bar{r}_2$	25
$\bar{r}^*$	28
$\bar{r}_2^*$	28
s	23
$\bar{t}_1$	23
$\bar{t}_2$	23
$\bar{t}_3$	23
$u_1$	2, 23
$u_2$	2, 23
$\bar{u}$	23
$\bar{u}_2$	28
(u)	38
x	29; 34
y	56
$y_1$	60
$y_2$	25; 60
$y_3$	25; 60

<u>Symbol</u>	<u>Page</u>
$y_j$	59
$z$	34; 53
$\Gamma_i$	106
$\Delta$	4, 45
$\Omega$	100
$\alpha_2$	27, 28
$\alpha^*$	110
$\gamma_{12}$	30
$\delta_1$	31
$\delta_2$	31
$(\delta)$	38
$\delta_{ij}$	106
$\epsilon_1$	29
$\epsilon_2$	29
$\eta$	110
$\theta$	2
$\lambda$	5, 56
$\lambda_I$	56
$\lambda_{II}$	56
$\lambda_{III}$	56
$\lambda_{11}$	62
$\lambda_{21}$	62
$\lambda_{31}$	62
$\lambda_{12}$	63
$\lambda_{22}$	63
$\lambda_{32}$	63
$\lambda_1$	65
$\lambda_2$	65
$\lambda_3$	65
$\lambda_n$	106
$\nu$	3, 37
$\xi$	110
$\pi$	58
$\rho$	3, 23

SymbolPage

$\sigma$	4, 45
$\sigma_{11}$	33
$\sigma_{12}$	33
$\sigma_i$	99
$\sigma_j$	99
$\tau$	109
$\phi$	64
$\psi$	2, 23
$\bar{\psi}$	23
$\omega$	59
$\omega_1$	61
$\omega_2$	61
$\omega_3$	61
$\omega_j$	61
$\omega_k$	61

## APPENDIX A

### GOVERNING EQUATIONS FOR THE DYNAMIC RESPONSE OF A CIRCULAR ARCH, INCLUDING BOTH SHEAR AND ROTARY INERTIA

Figure A1 shows a differential element of a circular arch. For the sign convention shown, with right-handed, orthogonal unit vectors  $\bar{t}_1$ ,  $\bar{t}_2$ , and  $\bar{t}_3$ ,

$$\bar{t}_3 = \bar{t}_1 \times \bar{t}_2 = \text{constant} \quad [A1]$$

$$\frac{d\bar{t}_1}{ds} = \bar{t}_1' = \frac{1}{R} \bar{t}_2 \quad [A2]$$

$$\frac{d\bar{t}_2}{ds} = \bar{t}_2' = -\frac{1}{R} \bar{t}_1 \quad [A3]$$

If we set

$$\bar{F} = F_1 \bar{t}_1 + F_2 \bar{t}_2 \quad (\text{internal force resultant}) \quad [A4]$$

$$\bar{M} = M \bar{t}_3 \quad (\text{internal moment resultant}) \quad [A5]$$

$$\bar{b} = b_1 \bar{t}_1 + b_2 \bar{t}_2 \quad (\text{external distributed load}) \quad [A6]$$

$$\bar{m} = m \bar{t}_3 \quad (\text{external distributed moment}) \quad [A7]$$

$$\bar{u} = u_1 \bar{t}_1 + u_2 \bar{t}_2 \quad (\text{centroidal displacement}) \quad [A8]$$

$$\bar{\psi} = \psi \bar{t}_3 \quad (\text{cross-section rotation}) \quad [A9]$$

then the translational equation of motion for the element is

$$-\bar{F} + \left[ \bar{F} + \frac{\partial \bar{F}}{\partial s} ds \right] + \bar{b} ds = \rho A ds \ddot{\bar{u}} \quad [A10]$$

where  $\rho$  is mass density, and  $A$  is cross-sectional area.



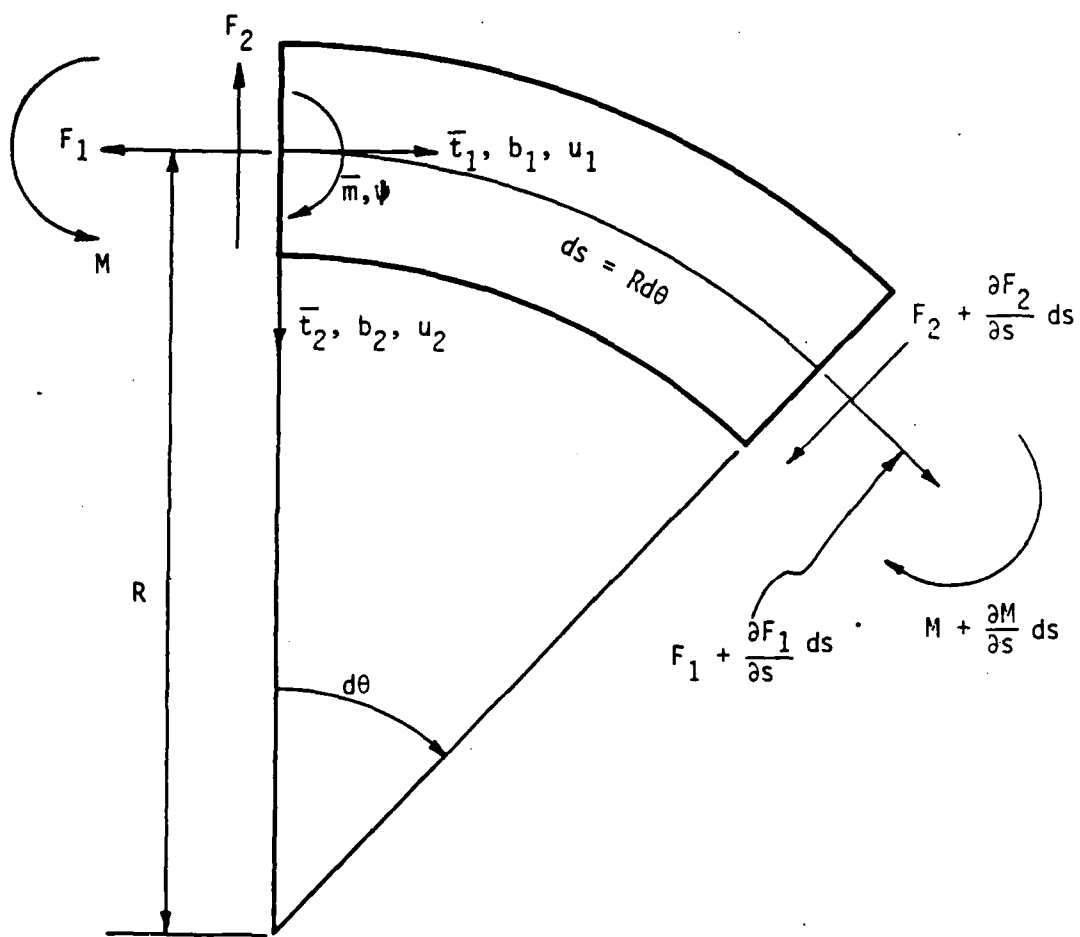


Figure A1. Differential Element of a Circular Arch.

or

$$\frac{\partial \bar{F}}{\partial s} + \bar{b} = \rho A \ddot{u} \quad [A11]$$

and the rotational equation of motion for the element is

$$-\bar{M} + \left[ \bar{M} + \frac{\partial \bar{M}}{\partial s} ds \right] + \bar{ds} \times \bar{F} + \bar{m} ds = \rho I ds \ddot{\psi} \quad [A12]$$

where  $I$  is the moment of inertia. Then

$$\frac{\partial \bar{M}}{\partial s} + \bar{t}_1 \times \bar{F} + \bar{m} = \rho I \ddot{\psi} \quad [A13]$$

Substitution of Equations A2 through A9 into Equations A11 and A13 yields

$$\frac{\partial F_1}{\partial s} - \frac{F_2}{R} + b_1 = \rho A \ddot{u}_1 \quad [A14]$$

$$\frac{\partial F_2}{\partial s} + \frac{F_1}{R} + b_2 = \rho A \ddot{u}_2 \quad [A15]$$

$$\frac{\partial M}{\partial s} + F_2 + m = \rho I \ddot{\psi} \quad [A16]$$

The next step in deriving the governing equations is to obtain expressions for the longitudinal extensional and shear strains at any point in an originally normal cross section. Consider a point  $Q$  in the undeformed arch shown in Figure A2, the position vector for which is denoted  $\bar{r}_2$  in Figure A3. From Figures A2 and A3 we have

$$\bar{r}_2 = \bar{r} + y_2 \bar{t}_2 + y_3 \bar{t}_3 \quad [A17]$$

so that in Figure A3,

$$\bar{PP}_1 = \frac{\partial \bar{r}}{\partial s} ds = \bar{t}_1 ds \quad [A18]$$

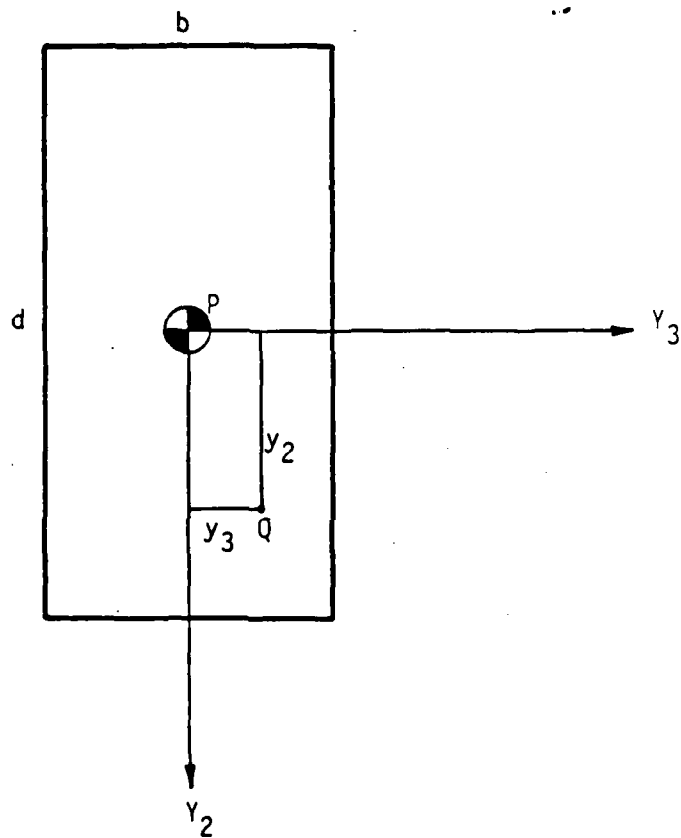


Figure A2. Normal Cross Section.

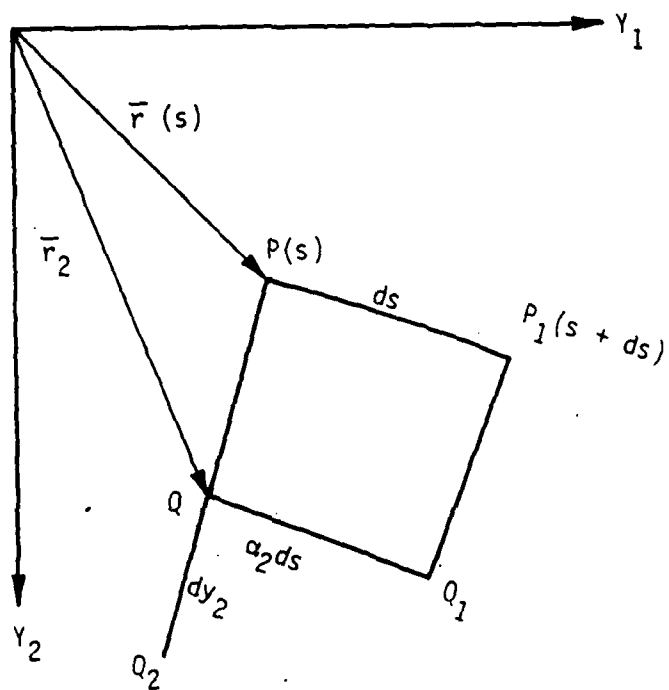


Figure A3. Differential Geometry in the Undeformed Arch.

$$\overline{QQ_1} = \frac{\partial \bar{r}_2}{\partial s} ds = \left[ 1 - \frac{y_2}{R} \right] d\bar{s}t_1 = \alpha_2 d\bar{s}t_1 \quad [A19]$$

$$\overline{QQ_2} = \frac{\partial \bar{r}_2}{\partial y_2} dy_2 = dy_2 \bar{t}_2 \quad [A20]$$

where

$$\alpha_2 = 1 - \frac{y_2}{R} \quad [A21]$$

The centroidal displacement vector of point P in the deformed arch is

$$\bar{u} = u_1 \bar{t}_1 + u_2 \bar{t}_2 \quad [A22]$$

and the assumption that plane cross sections normal to the centroidal axis before deformation remain plane, and normal to the 1-2 plane during deformation yields an expression for the displacement vector of point Q in the deformed arch

$$\bar{u}_2 = \bar{u} + \bar{\psi} \times y_2 \bar{t}_2 = (u_1 - \bar{\psi} y_2) \bar{t}_1 + u_2 \bar{t}_2 \quad [A23]$$

If  $P^*$ ,  $Q^*$ ,  $Q_1^*$ , and  $Q_2^*$  denote the displaced points P, Q,  $Q_1$ , and  $Q_2$  in the deformed arch, then the position vectors of points  $P^*$  and  $Q^*$  are

$$\bar{r}^* = \bar{r} + \bar{u} \quad [A24]$$

$$\bar{r}_2^* = \bar{r}_2 + \bar{u}_2 \quad [A25]$$

so that in the deformed arch, Equations A25 and A19 yield

$$\overline{Q^*Q_1^*} = \frac{\partial \bar{r}_2^*}{\partial s} ds = \left[ \alpha_2 \bar{t}_1 + \frac{\partial \bar{u}_2}{\partial s} \right] ds \quad [A26]$$

and Equations A25 and A20 yield

$$\overline{Q^* Q_2^*} = \frac{\partial \bar{r}_2^*}{\partial y_2} dy_2 = \left[ \bar{t}_2 + \frac{\partial \bar{u}_2}{\partial y_2} \right] dy_2 \quad [A27]$$

The components of strain are, therefore, assuming small displacements and rotations\*,

$$\begin{aligned} \epsilon_1 &= \sqrt{\frac{\overline{Q^* Q_1^*} \cdot \overline{Q^* Q_1^*}}{\overline{QQ_1} \cdot \overline{QQ_1}}} - 1 \\ &= \sqrt{\frac{\alpha_2^2 + 2\alpha_2 \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial s} + \frac{\partial \bar{u}_2}{\partial s} \cdot \frac{\partial \bar{u}_2}{\partial s}}{\alpha_2^2}} - 1 \\ &\approx \frac{1}{\alpha_2} \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial s} + \frac{1}{2\alpha_2^2} \frac{\partial \bar{u}_2}{\partial s} \cdot \frac{\partial \bar{u}_2}{\partial s} \end{aligned} \quad [A28]$$

$$\begin{aligned} \epsilon_2 &= \sqrt{\frac{\overline{Q^* Q_2^*} \cdot \overline{Q^* Q_2^*}}{\overline{QQ_2} \cdot \overline{QQ_2}}} - 1 \\ &= \sqrt{\frac{1 + 2\bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \frac{\partial \bar{u}_2}{\partial y_2} \cdot \frac{\partial \bar{u}_2}{\partial y_2}}{1}} - 1 \\ &\approx \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \frac{1}{2} \frac{\partial \bar{u}_2}{\partial y_2} \cdot \frac{\partial \bar{u}_2}{\partial y_2} \end{aligned} \quad [A29]$$

\*The surveyor's slope correction states that when  $x \ll 1$



$$\sqrt{1 + x^2} - 1 \approx \frac{x^2}{2}$$

$$\begin{aligned}
\gamma_{12} &= \frac{\overline{Q^* Q_1^*} \cdot \overline{Q^* Q_2^*}}{\left| \overline{Q^* Q_1^*} \right| \left| \overline{Q^* Q_2^*} \right|} \\
&= \frac{\alpha_2 \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial s} + \frac{\partial \bar{u}_2}{\partial s} \cdot \frac{\partial \bar{u}_2}{\partial y_2}}{\left| \alpha_2 \bar{t}_1 + \frac{\partial \bar{u}_2}{\partial s} \right| \left| \bar{t}_2 + \frac{\partial \bar{u}_2}{\partial y_2} \right|} \\
&\approx \frac{\alpha_2 \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial s} + \frac{\partial \bar{u}_2}{\partial s} \cdot \frac{\partial \bar{u}_2}{\partial y_2}}{\alpha_2 \cdot 1} \\
&\approx \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \frac{1}{\alpha_2} \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial s} + \frac{1}{\alpha_2} \frac{\partial \bar{u}_2}{\partial s} \cdot \frac{\partial \bar{u}_2}{\partial y_2} \quad [A30]
\end{aligned}$$

Assuming the displacement derivatives are small with respect to unity permits neglect of the nonlinear terms in Equations A28, A29, and A30, which leaves, using Equations A21 and A23,

$$\begin{aligned}
\epsilon_1 &= \frac{1}{\alpha_2} \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial s} \\
&= \frac{1}{1 - \frac{y_2}{R}} \bar{t}_1 \cdot \left[ \left[ \frac{\partial u_1}{\partial s} - y_2 \frac{\partial \psi}{\partial s} \right] \bar{t}_1 + (u_1 - y_2 \psi) \frac{\bar{t}_2}{R} + \frac{\partial u_2}{\partial s} \bar{t}_2 - u_2 \frac{\bar{t}_1}{R} \right] \\
&= \frac{1}{1 - \frac{y_2}{R}} \left[ \frac{\partial u_1}{\partial s} - y_2 \frac{\partial \psi}{\partial s} - \frac{u_2}{R} \right] \quad [A31]
\end{aligned}$$

$$\epsilon_2 = \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial y_2} = \bar{t}_2 \cdot (-\psi \bar{t}_1) = 0 \quad [A32]$$

$$\begin{aligned} \gamma_{12} &= \bar{t}_1 \cdot \frac{\partial \bar{u}_2}{\partial y_2} + \frac{1}{\alpha_2} \bar{t}_2 \cdot \frac{\partial \bar{u}_2}{\partial s} = \bar{t}_1 \cdot (-\psi \bar{t}_1) + \frac{1}{1 - \frac{y_2}{R}} \bar{t}_2 \\ &\cdot \left[ \left[ \frac{\partial u_1}{\partial s} - y_2 \frac{\partial \psi}{\partial s} \right] \bar{t}_1 + (u_1 - y_2 \psi) \frac{\bar{t}_2}{R} + \frac{\partial u_2}{\partial s} \bar{t}_2 - u_2 \frac{\bar{t}_1}{R} \right] \\ &= -\psi + \frac{1}{1 - \frac{y_2}{R}} \left[ \frac{u_1 - y_2 \psi}{R} + \frac{\partial u_2}{\partial s} \right] \\ &= \frac{1}{1 - \frac{y_2}{R}} \left[ -\psi + \frac{u_1}{R} + \frac{\partial u_2}{\partial s} \right] \quad [A33] \end{aligned}$$

Note that  $\gamma_{12}$  is positive when the dot product  $\overline{Q^* Q_1^*} \cdot \overline{Q^* Q_2^*}$  is positive, which means that the angle  $Q_1^* Q^* Q_2^*$  is less than  $\pi/2$ . When this is the case, the direction of the shear stresses on a deformed arch element are as shown in Figure A4.

If we now set

$$\delta_1 = (\epsilon_1)_{y_2=0} = \frac{\partial u_1}{\partial s} - \frac{u_2}{R} \quad [A34]$$

$$\delta_2 = (\gamma_{12})_{y_2=0} = -\psi + \frac{u_1}{R} + \frac{\partial u_2}{\partial s} \quad [A35]$$

$$k = \frac{\partial \psi}{\partial s} \quad [A36]$$

then Equations A31, A32, and A33 can be written in the form

$$\epsilon_1 = \frac{\delta_1 - y_2 k}{1 - \frac{y_2}{R}} \quad [A37]$$



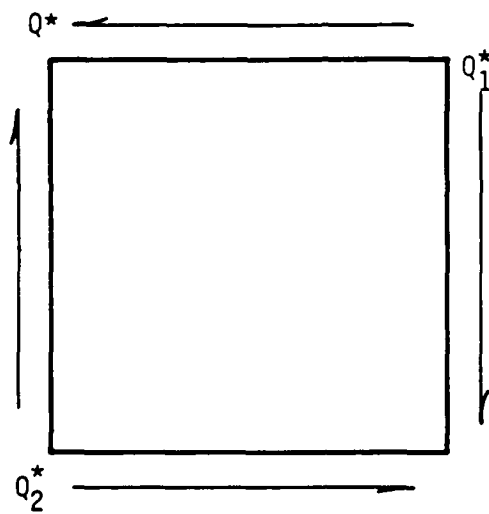


Figure A4. Shear Stress Directions  
for a Deformed Arch Element  
When the Angle  $Q_1^*Q^*Q_2^*$  is Acute.

$$\epsilon_2 = 0 \quad [A38]$$

$$\gamma_{12} = \frac{\delta_2}{1 - \frac{y_2}{R}} \quad [A39]$$

Assuming zero initial strain, the longitudinal normal and transverse shear stresses are

$$\sigma_{11} = E\epsilon_1 = \frac{E}{1 - \frac{y_2}{R}} (\delta_1 - y_2 k) \quad [A40]$$

$$\sigma_{12} = G\gamma_{12} = \frac{G}{1 - \frac{y_2}{R}} \delta_2 \quad [A41]$$

where  $E$  is Young's modulus, and  $G$  is the shear modulus, so that the stress resultants,  $F_1$ ,  $F_2$ , and  $M$  in Figure A1 are

$$F_1 = \iint \sigma_{11} dy_2 dy_3 = E\delta_1 \iint \frac{dy_2}{1 - \frac{y_2}{R}} dy_3 - Ek \iint \frac{y_2 dy_2}{1 - \frac{y_2}{R}} dy_3 \quad [A42]$$

$$F_2 = \iint \sigma_{12} dy_2 dy_3 = G\delta_2 \iint \frac{dy_2}{1 - \frac{y_2}{R}} dy_3 \quad [A43]$$

$$M = -\iint \sigma_{11} y_2 dy_2 dy_3 = -E\delta_1 \iint \frac{y_2 dy_2}{1 - \frac{y_2}{R}} dy_3 + Ek \iint \frac{y_2^2 dy_2}{1 - \frac{y_2}{R}} dy_3 \quad [A44]$$

The three integrals appearing in Equations A42, A43, and A44 can be expressed in terms of a single integral. Referring to Figure A2, we write

$$-\frac{d}{2} \left[ \frac{dy_2}{1 - \frac{y_2}{R}} \right] = -R \ln \left[ 1 - \frac{y_2}{R} \right] \Big|_{-\frac{d}{2}}^{\frac{d}{2}} = R \ln \left[ \frac{1 + \frac{d}{2R}}{1 - \frac{d}{2R}} \right] = d^* \quad [A45]$$

Now

$$\frac{1}{1-z} = \frac{1-z+z}{1-z} = 1 + z \left[ \frac{1}{1-z} \right] = 1 + z \left[ 1 + z \left[ \frac{1}{1-z} \right] \right]$$

so that

$$\frac{1}{1 - \frac{y_2}{R}} = 1 + \frac{\frac{y_2}{R}}{1 - \frac{y_2}{R}} = 1 + \frac{y_2}{R} + \frac{\left[ \frac{y_2}{R} \right]^2}{1 - \frac{y_2}{R}} \quad [A46]$$

Thus

$$-\frac{d}{2} \left[ \frac{y_2 dy_2}{1 - \frac{y_2}{R}} \right] = R(d^* - d) \quad [A47]$$

$$-\frac{d}{2} \left[ \frac{y_2^2 dy_2}{1 - \frac{y_2}{R}} \right] = R^2(d^* - d) \quad [A48]$$

Now

$$\ln \left[ \frac{1+x}{1-x} \right] = 2 \left[ x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots \right] = 2 \sum_{j=1}^{\infty} \frac{x^{2j-1}}{2j-1} \quad [A49]$$

so that if we set

$$\frac{d}{2R} = x \quad [A50]$$

then Equation A45 yields

$$\begin{aligned}
-\frac{d}{2} \left[ \frac{dy_2}{1 - \frac{y_2}{R}} \right] &= R \ln \left[ \frac{1+x}{1-x} \right] = \left[ \frac{R}{d} \ln \left[ \frac{1+x}{1-x} \right] \right] d \\
&= \frac{\ln \left[ \frac{1+x}{1-x} \right]}{2x} d = \left[ \frac{d^*}{d} \right] d = C_1 d
\end{aligned} \tag{A51}$$

where

$$C_1 = \frac{d^*}{d} = \frac{\ln \left[ \frac{1+x}{1-x} \right]}{2x} = 1 + \frac{x^2}{3} + \frac{x^4}{5} + \frac{x^6}{7} + \dots \tag{A52}$$

and therefore

$$(C_1)_{d/R=0} = 1 \tag{A53}$$

Equation A47 can be written in the form

$$-\frac{d}{2} \left[ \frac{y_2 dy_2}{1 - \frac{y_2}{R}} \right] = \left[ \frac{\frac{d^*}{d} - 1}{\frac{d}{R}} \right] d^2 = C_2 d^2 \tag{A54}$$

where

$$C_2 = \frac{\frac{d^*}{d} - 1}{\frac{d}{R}} = \frac{C_1 - 1}{2x} = \frac{x}{6} + \frac{x^3}{10} + \frac{x^5}{14} + \dots \tag{A55}$$

and therefore,

$$(C_2)_{d/R=0} = 0 \tag{A56}$$

Equation A48 can be written in the form

$$-\frac{d}{2} \left[ \frac{y_2^2 dy_2}{1 - \frac{y_2}{R}} \right] = 12 \left[ \frac{\left[ \frac{d}{R} \right]^2 - 1}{\left[ \frac{d}{R} \right]^2} \right] \frac{d^3}{12} = C_3 \frac{d^3}{12} \quad [\text{A57}]$$

where

$$C_3 = \frac{12 \left[ \left( \frac{d}{R} \right)^2 - 1 \right]}{\left[ \frac{d}{R} \right]^2} = \frac{3(C_1 - 1)}{x^2} = 1 + 3 \left[ \frac{x^2}{5} + \frac{x^4}{7} + \dots \right] \quad [\text{A58}]$$

and therefore,

$$(C_3)_{d/R=0} = 1 \quad [\text{A59}]$$

Values of  $C_1$ ,  $C_2$ , and  $C_3$  are tabulated below to four significant figures as a function of  $d/R$ .

$d/R$	$C_1$	$C_2$	$C_3$
0	1.000	0	1.000
.10	1.001	.0083	1.002
.20	1.003	.0168	1.006
.30	1.008	.0253	1.014
.40	1.014	.0342	1.025
.50	1.022	.0433	1.039

If we define the angular differential operator,  $D$ , to be

$$\frac{\partial}{\partial \theta} = D \quad [\text{A60}]$$

then Equations A14, A15, and A16 can be written in the form

$$\begin{bmatrix} \frac{D}{R} & -\frac{1}{R} & 0 \\ \frac{1}{R} & \frac{D}{R} & 0 \\ 0 & \frac{1}{r} & \frac{D}{R} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ \frac{M}{r} \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \frac{m}{r} \end{bmatrix} = \rho A \begin{bmatrix} u_1 \\ u_2 \\ r\psi \end{bmatrix} \quad [A61]$$

where

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{\frac{bd^3}{12}}{bd}} = \frac{d}{2\sqrt{3}} \quad [A62]$$

Equations A42, A43, and A44 can be written in the form

$$\begin{bmatrix} F_1 \\ F_2 \\ \frac{M}{r} \end{bmatrix} = EA \begin{bmatrix} C_1 & 0 & -C_2 \frac{d}{2r} \\ 0 & \frac{C_1}{2(1+\nu)} & 0 \\ -C_2 \frac{d}{2r} & 0 & C_3 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ rk \end{bmatrix} \quad [A63]$$

where  $\nu$  is Poisson's ratio. Equations A34, A35, and A36 can be written in the form

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ rk \end{bmatrix} = \begin{bmatrix} \frac{D}{R} & -\frac{1}{R} & 0 \\ \frac{1}{R} & \frac{D}{R} & -\frac{1}{r} \\ 0 & 0 & \frac{D}{R} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ r\psi \end{bmatrix} \quad [A64]$$

Equations A61, A63, and A64 are the governing equations for the dynamic response of a circular arch, including both shear and rotary inertia, with one exception. That exception is replacement of the constant  $C_1$  in the second of Equations A63 by the transverse shear coefficient,  $k'$ , to account for the fact that the arch cross section does not remain plane (Ref. 15).

Ref. 15. Oden, J. T. and E. A. Ripperger, Mechanics of Elastic Structures, 2nd Edition, McGraw-Hill, (1981).

## APPENDIX B

### MATRIX FORM OF THE GOVERNING EQUATIONS

Equations A61, A63, and A64 can be written in matrix form by setting

$$\underline{G} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{R}{r} & 0 \end{bmatrix} \quad [B1]$$

$$\underline{S} = \begin{bmatrix} C_1 & 0 & -C_2 \frac{d}{r} \\ 0 & \frac{k'}{2(1+\nu)} & 0 \\ -C_2 \frac{d}{r} & 0 & C_3 \end{bmatrix} \quad [B2]$$

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \\ \frac{M}{r} \end{bmatrix} \quad [B3]$$

$$\{\delta\} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ rk \end{bmatrix} \quad [B4]$$

$$\{u\} = \begin{bmatrix} u_1 \\ u_2 \\ r\psi \end{bmatrix} \quad [B5]$$

$$\{b\} = \begin{bmatrix} b_1 \\ b_2 \\ \frac{m}{r} \end{bmatrix} \quad [B6]$$

The equations of motion (A61) then take the form

$$\frac{1}{R} (D\mathbf{I} + \mathbf{G})\{\mathbf{F}\} + \{\mathbf{b}\} = \rho A \{\ddot{\mathbf{u}}\} \quad [\text{B7}]$$

where  $\mathbf{I}$  is the identity matrix,

the stress resultant-strain equations (A63) take the form

$$\{\mathbf{F}\} = E\mathbf{A}\mathbf{S} \{\boldsymbol{\delta}\} \quad [\text{B8}]$$

and the strain-displacement equations (A64) take the form

$$\{\boldsymbol{\delta}\} = \frac{1}{R} (D\mathbf{I} - \mathbf{G}^T)\{\mathbf{u}\} \quad [\text{B9}]$$

Substitution of Equations B8 and B9 into Equation B7 yields

$$\frac{EA}{R^2} (D\mathbf{I} + \mathbf{G})\mathbf{S}(D\mathbf{I} - \mathbf{G}^T)\{\mathbf{u}\} + \{\mathbf{b}\} = \rho A \{\ddot{\mathbf{u}}\} \quad [\text{B10}]$$

or

$$\frac{EA}{R^2} (D\mathbf{S} + \mathbf{GS})(D\mathbf{I} - \mathbf{G}^T)\{\mathbf{u}\} + \{\mathbf{b}\} = \rho A \{\ddot{\mathbf{u}}\}$$

or

$$\frac{EA}{R^2} \left[ D^2\mathbf{S} - D\mathbf{SG}^T + D\mathbf{GS} - \mathbf{GSG}^T \right] \{\mathbf{u}\} + \{\mathbf{b}\} = \rho A \{\ddot{\mathbf{u}}\}$$

or

$$\frac{EA}{R^2} \left[ D^2\mathbf{S} + D(\mathbf{GS} - \mathbf{SG}^T) - \mathbf{GSG}^T \right] \{\mathbf{u}\} + \{\mathbf{b}\} = \rho A \{\ddot{\mathbf{u}}\} \quad [\text{B11}]$$

The matrices in Equation B11 are computed below.



			$C_1$	$0$	$C_2 \frac{d}{r}$	[B12]
$\underline{S}$			$0$	$\frac{k'}{2(1+\nu)}$	$0$	
			$-C_2 \frac{d}{r}$	$0$	$C_3$	
$0$	$-1$	$0$	$0$	$-\frac{k'}{2(1+\nu)}$	$0$	
$1$	$0$	$0$	$C_1$	$0$	$C_2 \frac{d}{r}$	
$0$	$\frac{R}{r}$	$0$	$0$	$\frac{k'R}{2r(1+\nu)}$	$0$	
$\underline{G}$			$\underline{GS}$			

$$\underline{S}^T \underline{G}^T = \underline{SG}^T = \begin{bmatrix} 0 & C_1 & 0 \\ -\frac{k'}{2(1+\nu)} & 0 & \frac{k'R}{2r(1+\nu)} \\ 0 & -C_2 \frac{d}{r} & 0 \end{bmatrix} \quad [B13]$$

$$\underline{H} = \underline{GS} - \underline{SG}^T = \begin{bmatrix} 0 & -\frac{k'}{2(1+\nu)} - C_1 & 0 \\ \frac{k'}{2(1+\nu)} + C_1 & 0 & -\frac{k'R}{2r(1+\nu)} - C_2 \frac{d}{r} \\ 0 & \frac{k'R}{2r(1+\nu)} + C_2 \frac{d}{r} & 0 \end{bmatrix} \quad [B14]$$

Note that the matrix  $\underline{H}$  is antisymmetric.

$$\begin{array}{c}
 \underline{S} \underline{G}^T \\
 \begin{array}{|c|c|c|c|c|c|}
 \hline
 & & & 0 & C_1 & 0 \\
 \hline
 & & & -\frac{k'}{2(1+\nu)} & 0 & \frac{k'R}{2r(1+\nu)} \\
 \hline
 & & & 0 & -C_2 \frac{d}{r} & 0 \\
 \hline
 0 & -1 & 0 & \frac{k'}{2(1+\nu)} & 0 & -\frac{k'R}{2r(1+\nu)} \\
 \hline
 1 & 0 & 0 & 0 & C_1 & 0 \\
 \hline
 0 & \frac{R}{r} & 0 & -\frac{k'R}{2r(1+\nu)} & 0 & \frac{k'R^2}{2r^2(1+\nu)} \\
 \hline
 \end{array}
 \end{array}
 \quad \underline{G} \quad \underline{G} \underline{S} \underline{G}^T = \underline{K} \quad [B15]$$

If we set

$$c^2 = \frac{E}{\rho} \quad [B16]$$

then Equation B11 can be written in the form

$$\{\ddot{u}\} - \left[ \frac{c}{R} \right]^2 (D^2 \underline{S} + D \underline{H} - \underline{K}) \{u\} = \frac{1}{\rho A} \{b\} \quad [B17]$$

Equation B17 is the governing equation for the dynamic response of a circular arch, including both shear and rotary inertia, written in matrix form. The matrices  $\underline{S}$  and  $\underline{K}$  are symmetric, and the matrix  $\underline{H}$  is antisymmetric, i.e.,

$$\underline{S}^T = \underline{S} \quad [B18]$$

$$\underline{K}^T = \underline{K} \quad [B19]$$

$$\underline{H}^T = -\underline{H} \quad [B20]$$

Note also that Equation A62 yields

$$\frac{d}{r} = 2\sqrt{3} \quad [B21]$$

so that we can write

$$c_2 \frac{d}{r} = 2\sqrt{3} c_2$$

[B22]

## APPENDIX C

### UNCOUPLING THE EQUATIONS OF MOTION

For free vibration, we set

$$\{b\} = \{0\} \quad [C1]$$

in Equation B17, and obtain

$$\ddot{\{u\}} - \left[ \frac{C}{R} \right]^2 (D^2 \underline{S} + D\underline{H} - \underline{K}) \{u\} = \{0\} \quad [C2]$$

Assuming a separable solution, we set

$$\{u\} = T\{U\} \quad [C3]$$

where  $T$  is a function of time only, and

$$\{U\} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad [C4]$$

where  $X$ ,  $Y$ , and  $Z$  are functions of  $\theta$  only.

Substitution of Equation C3 into Equation C2 yields

$$T\{U\} - \left[ \frac{C}{R} \right]^2 T(D^2 \underline{S} + D\underline{H} - \underline{K})\{U\} = \{0\} \quad [C5]$$

or

$$T\{U\} - \left[ \frac{C}{R} \right]^2 T\underline{L}\{U\} = \{0\} \quad [C6]$$

where  $\underline{L}$  is the linear, spatial, matrix differential operator,

$$\underline{L} = D^2 \underline{S} + D\underline{H} - \underline{K} \quad [C7]$$

It is convenient to define a diagonal matrix containing the elements  $X$ ,  $Y$ , and  $Z$ ,

$$\bar{U} = \begin{bmatrix} X & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{bmatrix} \quad [C8]$$

(The notation  $\bar{\phantom{U}}$  denotes a diagonal matrix.)

as well as the identity vector

$$\{m\} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad [C9]$$

so that

$$\bar{U}^{-1} \{U\} = \{m\} \quad [C10]$$

and

$$\bar{U} \{m\} = \{U\} \quad [C11]$$

Premultiplication of Equation C6 by  $\bar{U}^{-1}$  yields

$$\ddot{T}\{m\} - \left[\frac{C}{R}\right]^2 \bar{U}^{-1} L\{U\} = \{0\} \quad [C12]$$

and division by  $T$  yields

$$\frac{\ddot{T}}{T}\{m\} = \left[\frac{C}{R}\right]^2 \bar{U}^{-1} L\{U\} \quad [C13]$$

Since the LHS of Equation C13 is a function of time only, and the RHS is a function of  $\theta$  only, for arbitrary values of  $T$  and  $\theta$ , both sides must be constant, so that

$$\frac{\ddot{T}}{T}\{m\} = \left[\frac{C}{R}\right]^2 \bar{U}^{-1} L\{U\} = -p^2 \{m\} \quad [C14]$$

and therefore

$$\ddot{T} + p^2 T = 0 \quad [C15]$$

The function  $T$  is therefore harmonic. Premultiplication of the last two parts of Equation C14 by  $\begin{bmatrix} \bar{U} \end{bmatrix}$  now yields

$$\left[ \frac{c}{R} \right]^2 \underline{L}\{U\} = -p^2\{U\}$$

or

$$\underline{L}\{U\} = - \frac{p^2}{\left[ \frac{c}{R} \right]^2} \{U\} = -\sigma^2\{U\} \quad [C16]$$

where

$$\sigma = \frac{p}{\left[ \frac{c}{R} \right]} \quad [C17]$$

Equation C16 can be written in the form

$$(\underline{L} + \sigma^2 \underline{I})\{U\} = \{0\} \quad [C18]$$

The linear differential operator coefficient matrix  $(\underline{L} + \sigma^2 \underline{I})$  in Equation C18 is coupled, but the system of ordinary differential equations can be uncoupled by premultiplying by the adjoint of the coefficient matrix.

$$(\underline{L} + \sigma^2 \underline{I})^* (\underline{L} + \sigma^2 \underline{I})\{U\} = \Delta \underline{I}\{U\} = \Delta\{U\} = \{0\} \quad [C19]$$

where  $\Delta$  is the determinant of the coefficient matrix in Equation C18.

$$\Delta = \left| \underline{L} + \sigma^2 \underline{I} \right| \quad [C20]$$

Equation C19 shows that X, Y, and Z each satisfies the same linear, ordinary differential equation:

$$\Delta X = 0 \quad [C21]$$

$$\Delta Y = 0 \quad [C22]$$

$$\Delta Z = 0 \quad [C23]$$

The system of equations is thus uncoupled.

# APPENDIX D

## EXPANSION OF THE CHARACTERISTIC DETERMINANT

In Equations C21, C22, and C23, the linear, ordinary, differential operator  $\Delta$  represents the expansion of a determinant.

$$\Delta = \left| \underline{L} + \sigma^2 \underline{I} \right| \quad [C20 \text{ bis}]$$

where

$$\underline{L} = D^2 \underline{S} + D \underline{H} - \underline{K} \quad [C7 \text{ bis}]$$

and

$$\underline{S} = \begin{bmatrix} C_1 & 0 & -2\sqrt{3}C_2 \\ 0 & \frac{k'}{2(1+\nu)} & 0 \\ -2\sqrt{3}C_2 & 0 & C_3 \end{bmatrix} \quad [D1]$$

$$\underline{H} = \begin{bmatrix} 0 & -\frac{k'}{2(1+\nu)} - C_1 & 0 \\ \frac{k'}{2(1+\nu)} + C_1 & 0 & -\frac{k'R}{2r(1+\nu)} - 2\sqrt{3}C_2 \\ 0 & \frac{k'R}{2(1+\nu)} + 2\sqrt{3}C_2 & 0 \end{bmatrix} \quad [D2]$$

$$\underline{K} = \begin{bmatrix} \frac{k'}{2(1+\nu)} & 0 & -\frac{k'R}{2r(1+\nu)} \\ 0 & C_1 & 0 \\ -\frac{k'R}{2r(1+\nu)} & 0 & \frac{k'R^2}{2r^2(1+\nu)} \end{bmatrix} \quad [B15 \text{ bis}]$$

To facilitate the expansion of  $\Delta$ , let

$$C_1 = A \quad [D3]$$



$$\frac{k'}{2(1+\nu)} = B \quad [04]$$

$$C_3 = C \quad [05]$$

$$2\sqrt{3}C_2 = E \quad [06]$$

$$\frac{R}{r} = F \quad [07]$$

Then Equations C7, D1, D2, and B15 yield

$$\begin{aligned} \underline{L} &= D^2 \begin{bmatrix} A & 0 & -E \\ 0 & B & 0 \\ -E & 0 & C \end{bmatrix} + D \begin{bmatrix} 0 & -(A+B) & 0 \\ (A+B) & 0 & -(BF+E) \\ 0 & (BF+E) & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} B & 0 & -BF \\ 0 & A & 0 \\ -BF & 0 & BF^2 \end{bmatrix} \\ &= \begin{bmatrix} AD^2-B & -(A+B)D & -ED^2+BF \\ (A+B)D & BD^2-A & -(BF+E)D \\ -ED^2+BF & (BF+E)D & CD^2-BF^2 \end{bmatrix} \quad [08] \end{aligned}$$

so that Equation C20 takes the form

$$\Delta = \begin{vmatrix} (AD^2-B)+\sigma^2 & -(A+B)D & -ED^2+BF \\ (A+B)D & (BD^2-A)+\sigma^2 & -(BF+E)D \\ -ED^2+BF & (BF+E)D & (CD^2-BF^2)+\sigma^2 \end{vmatrix} \quad [09]$$

Equation D9 is very similar to the determinant which arises in calculating principal stresses, so that the expansion of Equation D9 is known to take the form

$$\Delta = \sigma^6 + I_1 \sigma^4 + I_2 \sigma^2 + I_3 \quad [D10]$$

where

$$I_1 = \text{Tr} (\underline{L}) \quad [D11]$$

$$I_2 = \begin{vmatrix} AD^2-B & -(A+B)D \\ (A+B)D & BD^2-A \end{vmatrix} + \begin{vmatrix} BD^2-A & -(BF+E)D \\ (BF+E)D & CD^2-BF^2 \end{vmatrix} \\ + \begin{vmatrix} CD^2-BF^2 & -ED^2+BF \\ -ED^2+BF & AD^2-B \end{vmatrix} \quad [D12]$$

$$I_3 = |\underline{L}| = \begin{vmatrix} AD^2-B & -(A+B)D & -ED^2+BF \\ (A+B)D & BD^2-A & -(BF+E)D \\ -ED^2+BF & (BF+E)D & CD^2-BF^2 \end{vmatrix} \quad [D13]$$

Equations D11, D12, and D13 yield

$$I_1 = (A + B + C^2)D - (A + B + BF^2) \quad [D14]$$

$$I_2 = (AB + BC + AC - E^2)D^4 \\ + (2AB - AC + 4BEF + E^2 - BC - ABF^2)D^2 + (AB + ABF^2) \quad [D15]$$

$$I_3 = (ABC - BE^2)(D^2 + 1)^2 D^2 \quad [D16]$$

Substitution of Equations D14, D15, and D16 into Equation D10 yields

$$\Delta = \sigma^6 + \left[ (A + B + C)D^2 - (A + B + BF^2) \right] \sigma^4 \\ + \left[ (AB + BC + AC - E^2)D^4 + (2AB - AC + 4BEF + E^2 - BC - ABF^2)D^2 \right. \\ \left. + (AB + ABF^2) \right] \sigma^2 \\ + \left[ (ABC - BE^2)D^6 + 2(ABC - BE^2)D^4 + (ABC - BE^2)D^2 \right] \quad [D17]$$

Finally,  $\Delta$  can be expressed as a polynomial in the operator  $D$ .

$$\Delta = (ABC - BE^2)D^6 + \left[ (AB + BC + AC - E^2)\sigma^2 + 2(ABC - BE^2) \right] D^4 \\ + \left[ (A + B + C)\sigma^4 + (2AB - AC + 4BEF + E^2 - BC - ABF^2)\sigma^2 \right. \\ \left. + (ABC - BE^2) \right] D^2 + \left[ \sigma^6 - (A + B + BF^2)\sigma^4 + (AB + ABF^2)\sigma^2 \right] \quad [D18]$$

All the coefficients in Equation D18 can be expressed in terms of the invariants of the matrices  $\underline{S}$ ,  $\underline{H}$ ,  $\underline{K}$ , and  $\underline{S} - \underline{K}$ , where  $I_j(\underline{x})$  is the  $j^{\text{th}}$  invariant of the matrix,  $\underline{x}$ .

$$I_1(\underline{S}) = A + B + C \quad [D19]$$

$$I_2(\underline{S}) = AB + BC + AC - E^2 \quad [D20]$$

$$I_3(\underline{S}) = ABC - BE^2 \quad [D21]$$

$$I_1(\underline{H}) = 0 \quad [D22]$$

$$I_2(\underline{H}) = A^2 + 2AB + B^2 + B^2F^2 + 2BEF + E^2 \quad [D23]$$

$$I_3(\underline{H}) = 0 \quad [D24]$$

$$I_1(\underline{K}) = A + B + BF^2 \quad [D25]$$

$$I_2(\underline{K}) = AB + ABF^2 \quad [D26]$$

$$I_3(\underline{K}) = 0 \quad [D27]$$

$$\underline{S} - \underline{K} = \begin{bmatrix} A-B & 0 & BF-E \\ 0 & B-A & 0 \\ BF-E & 0 & C-BF^2 \end{bmatrix} \quad [D28]$$

$$I_2(\underline{S} - \underline{K}) = -A^2 + 2AB - B^2 - B^2F^2 + 2BEF - E^2 \quad [D29]$$

$$I_2(\underline{S} - \underline{K}) = I_2(\underline{S}) - I_2(\underline{K}) + I_2(\underline{H}) \\ = 2AB - AC + 4BEF + E^2 - BC - ABF^2 \quad [D30]$$

Thus, Equation D18 can be written in the form

$$\begin{aligned} \Delta = & I_3(\underline{S})D^6 + \left[ I_2(\underline{S})\sigma^2 + 2I_3(\underline{S}) \right] D^4 \\ & + \left[ I_1(\underline{S})\sigma^4 + \left[ I_2(\underline{S} - \underline{K}) - I_2(\underline{S}) - I_2(\underline{K}) + I_2(\underline{H}) \right] \sigma^2 + I_3(\underline{S}) \right] D^2 \\ & + \left[ \sigma^6 - I_1(\underline{K})\sigma^4 + I_2(\underline{K})\sigma^2 \right] \end{aligned} \quad [D31]$$

Substitution of Equations D3 - D7 into Equations D19, D20, D21, D25, D26, and D30 yields

$$I_1(\underline{S}) = \left[ C_1 + C_3 \right] + \frac{k'}{2(1 + \nu)} \quad [D32]$$

$$I_2(\underline{S}) = \frac{k'}{2(1 + \nu)} \left[ C_1 + C_3 \right] + \left[ C_1 C_3 - 12C_2^2 \right] \quad [D33]$$

$$I_3(\underline{S}) = \frac{k'}{2(1 + \nu)} \left[ C_1 C_3 - 12C_2^2 \right] \quad [D34]$$

$$I_1(\underline{K}) = C_1 + \frac{k'}{2(1 + \nu)} \left[ 1 + \left[ \frac{R}{r} \right]^2 \right] \quad [D35]$$

$$I_2(\underline{K}) = \frac{C_1 k'}{2(1 + \nu)} \left[ 1 + \left[ \frac{R}{r} \right]^2 \right] \quad [D36]$$

$$\begin{aligned} I_2(\underline{S} - \underline{K}) - I_2(\underline{S}) - I_2(\underline{K}) + I_2(\underline{H}) \\ = \frac{k'}{2(1 + \nu)} \left[ 2C_1 - C_3 + 8\sqrt{3}C_2 \left[ \frac{R}{r} \right] - C_1 \left[ \frac{R}{r} \right]^2 \right] \\ - \left[ C_1 C_3 - 12C_2^2 \right] \end{aligned} \quad [D37]$$

Therefore, if we set

$$a = \frac{I_2(\underline{S})}{I_3(\underline{S})} \sigma^2 + 2 \quad [D38]$$

$$b = \frac{I_1(\underline{S})\sigma^4 + \left[ I_2(\underline{S} - \underline{K}) - I_2(\underline{S}) - I_2(\underline{K}) + I_2(\underline{H}) \right] \sigma^2}{I_3(\underline{S})} + 1 \quad [D39]$$

$$c = \frac{\sigma^6 - I_1(K)\sigma^4 + I_2(K)\sigma^2}{I_3(S)} \quad [D40]$$

then Equation D31 can be written in the form

$$\Delta = I_3(S) (D^6 + aD^4 + bD^2 + c) \quad [D41]$$

Inspection of Equations A52, A55, and A58 shows that  $I_3(S)$ , as defined by Equation D34, is always positive.

Referring to Equations D32-D37, we see that

$$\frac{I_2(S)}{I_3(S)} = \frac{C_1 + C_3}{C_1 C_3 - 12C_2^2} + \frac{2(1 + \nu)}{k'} \quad [D42]$$

$$\frac{I_1(S)}{I_3(S)} = \frac{2(1 + \nu)}{k'} \left[ \frac{C_1 + C_3}{C_1 C_3 - 12C_2^2} \right] + \frac{1}{C_1 C_3 - 12C_2^2} \quad [D43]$$

$$\begin{aligned} \frac{I_2(S - K) - I_2(S) - I_2(K) + I_3(H)}{I_3(S)} \\ = \frac{2C_1 - C_3 + 8\sqrt{3}C_2 \left[ \frac{R}{r} \right] - C_1 \left[ \frac{R}{r} \right]^2}{C_1 C_3 - 12C_2^2} - \frac{2(1 + \nu)}{k'} \end{aligned} \quad [D44]$$

$$\frac{I_1(K)}{I_3(S)} = \frac{2(1 + \nu)}{k'} \left[ \frac{C_1}{C_1 C_3 - 12C_2^2} \right] + \frac{1 + \left[ \frac{R}{r} \right]^2}{C_1 C_3 - 12C_2^2} \quad [D45]$$

$$\frac{I_2(K)}{I_3(S)} = \frac{C_1}{C_1 C_3 - 12C_2^2} \left[ 1 + \left[ \frac{R}{r} \right]^2 \right] \quad [D46]$$

and if we set

$$\frac{1}{C_1 C_3 - 12 C_2^2} = K_0 \quad [D47]$$

$$\frac{C_1}{K_0} = K_1 \quad [D48]$$

$$\frac{C_2}{K_0} = K_2 \quad [D49]$$

$$\frac{C_3}{K_0} = K_3 \quad [D50]$$

$$\frac{2(1 + \nu)}{K'} = n \quad [D51]$$

$$\frac{R}{r} = z \quad [D52]$$

then Equations D38-D40 can be written in the form

$$a = (K_1 + K_3 + n)\sigma^2 + 2 \quad [D53]$$

$$b = \left[ n(K_1 + K_3) + K_0 \right] \sigma^4 + \left[ K_1(2 - z^2) + 8\sqrt{3}K_2z - K_3 - n \right] \sigma^2 + 1 \quad [D54]$$

$$c = nK_0\sigma^6 - \left[ nK_1 + K_0(1 + z^2) \right] \sigma^4 + K_1(1 + z^2)\sigma^2 \quad [D55]$$

The coefficients  $K_0$ ,  $K_1$ ,  $K_2$ , and  $K_3$  are functions of  $d/R$ .

d/R	$C_1$	$C_2$	$C_3$	$K_0$	$K_1$	$K_2$	$K_3$
0	1.000	0	1.000	1.000	1.000	0	1.000
.10	1.001	.0083	1.002	0.998	0.999	0.0083	1.000
.20	1.003	.0168	1.006	0.994	0.997	0.0167	1.000
.30	1.008	.0253	1.014	0.986	0.994	0.0249	1.000
.40	1.014	.0342	1.025	0.975	0.989	0.0334	1.000
.50	1.022	.0433	1.039	0.962	0.983	0.0417	1.000

Using the commonly accepted value

$$k' = \frac{5}{6} \approx \frac{\pi^2}{12} \quad [D56]$$

Equation D51 yields

$$n = \frac{12(1 + \nu)}{5} \quad [D57]$$

$\nu$	$n$
0.10	2.64
0.15	2.76
0.20	2.88
0.25	3.00
0.30	3.12
0.35	3.24

Assuming, based on the values tabulated on the previous page, that

$$K_0 = K_1 = K_3 = 1$$

$$K_2 = 0$$

Equations D53, D54, and D55 reduce to

$$a = (2 + n)\sigma^2 + 2 \quad [D58]$$

$$b = (2n + 1)\sigma^4 - (z^2 + n - 1)\sigma^2 + 1 \quad [D59]$$

$$c = [n\sigma^4 - (z^2 + n + 1)\sigma^2 + (1 + z^2)]\sigma^2 \quad [D60]$$

and if we assume that

$$n = 3$$

$$z^2 + n \pm 1 \approx z^2$$

Equations D58-D60 reduce to

$$a = 5\sigma^2 + 2 \quad [D61]$$

$$b = 7\sigma^4 - z^2\sigma^2 + 1 \quad [D62]$$

$$c = 3\sigma^4 - z^2\sigma^2 + z^2 \quad [D63]$$

Equations D53-D55 and D58-D63 show the relation between structural parameters and frequency, and the coefficients of the characteristic equation, D41, under various assumed (approximate) conditions.



## APPENDIX E

### GENERAL SOLUTION OF THE CHARACTERISTIC DIFFERENTIAL EQUATION

The constant  $I_3$  ( $\underline{S}$ ) in Equation D41 is nonzero, and can therefore be ignored. We therefore seek the general solution of the homogeneous, linear, ordinary differential equation

$$LX = 0 \quad [E1]$$

where

$$L = D^6 + aD^4 + bD^2 + c \quad [E2]$$

$$D = \frac{d}{d\theta} \quad [E3]$$

The solution of Equation E1 is assumed to be of the form

$$X = Ke^{\lambda\theta} \quad [E4]$$

and substitution of Equation E4 into Equation E1 yields

$$(\lambda^6 + a\lambda^4 + b\lambda^2 + c)Ke^{\lambda\theta} = 0 \quad [E5]$$

If a nontrivial solution exists, it must be that

$$\lambda^6 + a\lambda^4 + b\lambda^2 + c = 0 \quad [E6]$$

Equation E6 is a cubic equation in  $\lambda^2$ , and therefore has three roots:

$$\lambda_I^2, \lambda_{II}^2, \text{ and } \lambda_{III}^2$$

Equation E6 can therefore be written in the form

$$\left[ \lambda^2 - \lambda_I^2 \right] \left[ \lambda^2 - \lambda_{II}^2 \right] \left[ \lambda^2 - \lambda_{III}^2 \right] = \lambda^6 + a\lambda^4 + b\lambda^2 + c = 0 \quad [E7]$$

Comparison of Equations E6 and E7 shows that

$$a = - \left[ \lambda_I^2 + \lambda_{II}^2 + \lambda_{III}^2 \right] \quad [E8]$$

$$b = \left[ \lambda_I^2 \lambda_{II}^2 + \lambda_{II}^2 \lambda_{III}^2 + \lambda_{III}^2 \lambda_I^2 \right] \quad [E9]$$

$$c = - \lambda_I^2 \lambda_{II}^2 \lambda_{III}^2 \quad [E10]$$

The classic algebraic solution of a cubic equation (E6) begins by setting

$$\lambda^2 = y - \frac{a}{3} \quad [E11]$$

in order to obtain the reduced cubic equation

$$y^3 - Py - Q = 0 \quad [E12]$$

where

$$P = \frac{a^2}{3} - b \quad [E13]$$

$$Q = \frac{ab}{3} - \frac{2a^3}{27} - c \quad [E14]$$

The solution of Equation E12 is obtained by assuming it to be of the form

$$y = A + B \quad [E15]$$

which leads to the requirement that

$$AB = \frac{P}{3} \quad [E16]$$

and

$$A^3 + B^3 = Q \quad [E17]$$

Equations E16 and E17 generate a quadratic equation, the roots of which are

$$A^3 = \frac{Q}{2} + \sqrt{\left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3} \quad [E18]$$

$$B^3 = \frac{Q}{2} - \sqrt{\left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3} \quad [E19]$$

and if we set

$$R = \left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3 \quad [E20]$$

then Equations E18 and E19 can be written in the form

$$A^3 = \frac{Q}{2} + \sqrt{R} \quad [E21]$$

$$B^3 = \frac{Q}{2} - \sqrt{R} \quad [E22]$$

The nature of the roots  $A^3$  and  $B^3$  is determined by the relative values of  $P$  and  $Q$ .

If

$$\left[\frac{P}{3}\right]^3 \leq \left[\frac{Q}{2}\right]^2 \quad [E23]$$

then the quantity  $R$  defined by Equation E20 is nonnegative, so that  $\sqrt{R}$ ,  $A^3$ , and  $B^3$  are real. Then if we set

$$A = \left[\frac{Q}{2} + \sqrt{R}\right]^{1/3} \quad [E24]$$

$$B = \left[\frac{Q}{2} - \sqrt{R}\right]^{1/3} \quad [E25]$$

and then, using  $\pi$  as the constant, 3.14159..., set

$$A_1 = A \quad [E26]$$

$$A_2 = Ae^{i\frac{2\pi}{3}} \quad [E27]$$

$$A_3 = Ae^{-i\frac{2\pi}{3}} \quad [E28]$$

$$B_1 = B \quad [E29]$$

$$B_2 = Be^{i\frac{2\pi}{3}} \quad [E30]$$

$$B_3 = B e^{-i\frac{2\pi}{3}} \quad [E31]$$

then Equations E16 and E17 will be satisfied by setting

$$y_j = A_j + B_j \quad (j = 1, 2, 3) \quad [E32]$$

If, on the other hand,

$$\left[\frac{P}{3}\right]^3 > \left[\frac{Q}{2}\right]^2 \quad [E33]$$

then the quantity  $R$  defined by Equation E20 is negative, so that  $\sqrt{R}$  is imaginary and  $A^3$  and  $B^3$  are complex. Then if we set

$$A^3 = \frac{Q}{2} + i \sqrt{\left[\frac{P}{3}\right]^3 - \left[\frac{Q}{2}\right]^2} \quad [E34]$$

$$B^3 = \frac{Q}{2} - i \sqrt{\left[\frac{P}{3}\right]^3 - \left[\frac{Q}{2}\right]^2} \quad [E35]$$

then we can show the complex quantities  $A^3$  and  $B^3$  in an Argand diagram (Figure E). Note that Equation E33 guarantees that  $P$  is positive. Figure E yields

$$A^3 = \left[\frac{P}{3}\right]^{3/2} e^{i3\omega} \quad [E36]$$

$$B^3 = \left[\frac{P}{3}\right]^{3/2} e^{-i3\omega} \quad [E37]$$

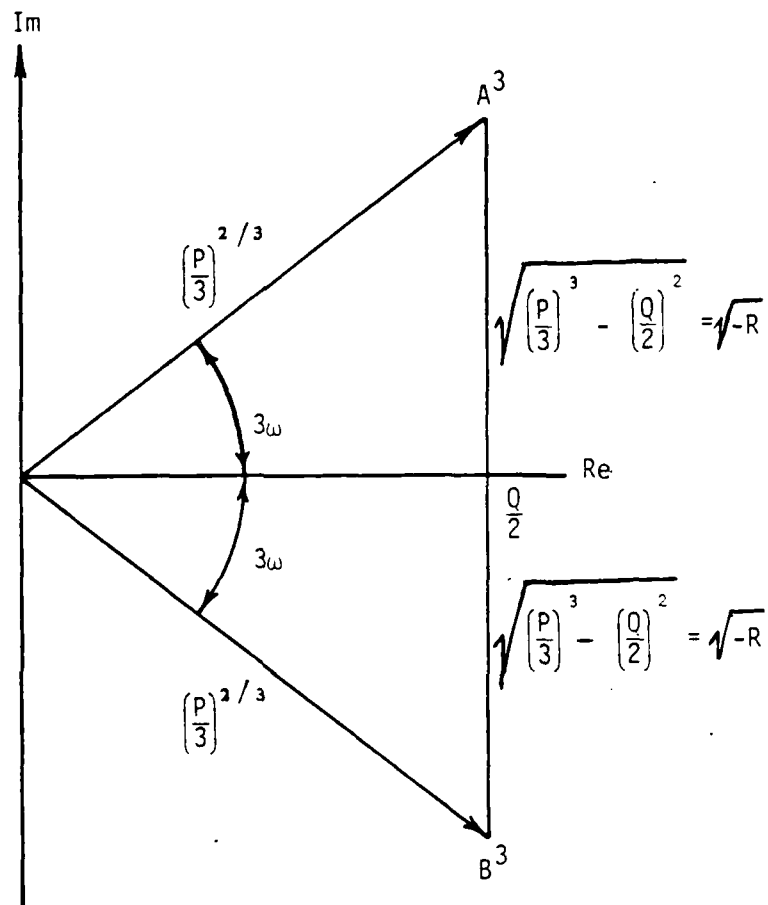


Figure E. Argand Diagram Showing  $A^3$  and  $B^3$   
When  $R < 0$ .

where

$$\cos 3\omega = \frac{\frac{Q}{2}}{\left[\frac{P}{3}\right]^{3/2}} \quad (0 \leq 3\omega \leq \pi) \quad [\text{E38}]$$

If we set

$$\omega_1 = \omega \quad [\text{E39}]$$

$$\omega_2 = \omega - \frac{2\pi}{3} \quad [\text{E40}]$$

$$\omega_3 = \omega + \frac{2\pi}{3} \quad [\text{E41}]$$

and then set

$$A_j = \left[\frac{P}{3}\right]^{1/2} e^{i\omega_j} \quad (j = 1, 2, 3) \quad [\text{E42}]$$

$$B_k = \left[\frac{P}{3}\right]^{1/2} e^{i\omega_k} \quad (j = 1, 2, 3) \quad [\text{E43}]$$

then Equations E16 and E17 will be satisfied by setting

$$y_j = A_j + B_j = 2\sqrt{\frac{P}{3}} \cos \omega_j \quad (j = 1, 2, 3) \quad [\text{E44}]$$

Finally, in keeping with Equation E11, we write

$$\lambda_I^2 = y_1 - \frac{a}{3} \quad [\text{E45a}]$$

$$\lambda_{II}^2 = y_2 - \frac{a}{3} \quad [\text{E45b}]$$

$$\lambda_{III}^2 = y_3 - \frac{a}{3} \quad [\text{E45c}]$$

We now need to determine whether the roots  $\lambda_I^2$ ,  $\lambda_{II}^2$ , and  $\lambda_{III}^2$  are real or complex, and if real, whether they are positive or negative.

## APPENDIX F

### CHARACTERISTIC MODE FUNCTIONS

The exact nature of the three roots,  $\lambda_I^2$ ,  $\lambda_{II}^2$ , and  $\lambda_{III}^2$ , defined by Equations E45, determines the forms of the arch mode shapes. The three roots are determined by the coefficients a, b, and c, which appear in Equation E2 and are defined by Equations D38-D40, D53-D55, D58-D60, or D61-D63.

First, note that Equation D53 guarantees the coefficient "a" will always be positive.

$$a > 0 \quad [F1]$$

Now consider Equation E12:

$$y^3 - Py - Q = 0 \quad [E12 \text{ bis}]$$

The various root combinations for Equation E12, and the arch mode shapes associated with each are discussed in detail below.

#### Case 1: $P = Q = 0$

In Case 1, Equation E12 reduces to

$$y^3 = 0 \quad [F2]$$

and the roots are

$$y_1 = y_2 = y_3 = 0 \quad [F3]$$

Therefore Equations E45 yield

$$\lambda_I^2 = \lambda_{II}^2 = \lambda_{III}^2 = -\frac{a}{3} \quad [F4]$$

so that

$$\lambda_{11} = \lambda_{21} = \lambda_{31} = i\sqrt{\frac{a}{3}} \quad [F5]$$

$$\lambda_{12} = \lambda_{22} = \lambda_{32} = -i\sqrt{\frac{a}{3}} \quad [F6]$$

If we set

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda = \sqrt{\frac{a}{3}} \quad [F7]$$

then because of the repeated nature of the roots, the mode functions are

$$w_1 = \cos \lambda \theta \quad [F8]$$

$$w_2 = \sin \lambda \theta \quad [F9]$$

$$w_3 = \theta \cos \lambda \theta \quad [F10]$$

$$w_4 = \theta \sin \lambda \theta \quad [F11]$$

$$w_5 = \theta^2 \cos \lambda \theta \quad [F12]$$

$$w_6 = \theta^2 \sin \lambda \theta \quad [F13]$$

Case 2:  $P = 0; Q > 0$

In Case 2, Equation E12 reduces to

$$y^3 - |Q| = 0 \quad [F14]$$

so that

$$y^3 = |Q| \quad [F15]$$

Now let

$$r = |Q|^{1/3} > 0 \quad [F16]$$

Then

$$y_1 = r \quad [F17]$$

$$y_2 = re^{i\frac{2\pi}{3}} = \frac{r}{2}(-1 + i\sqrt{3}) \quad [F18]$$



$$y_3 = re^{-i\frac{2\pi}{3}} = -\frac{r}{2}(1 + i\sqrt{3}) \quad [\text{F19}]$$

and Equations E45 yield

$$\lambda_I^2 = r - \frac{a}{3} \quad [\text{F20}]$$

$$\lambda_{II}^2 = -\left[\frac{r}{2} + \frac{a}{3}\right] + i\frac{r\sqrt{3}}{2} = Se^{i\phi} \quad [\text{F21}]$$

$$\lambda_{III}^2 = -\left[\frac{r}{2} + \frac{a}{3}\right] - i\frac{r\sqrt{3}}{2} = Se^{-i\phi} \quad [\text{F22}]$$

where

$$S^2 = \left[\frac{r}{2} + \frac{a}{3}\right]^2 + \frac{3r^2}{4} = r^2 + \frac{ra}{3} + \frac{a^2}{9} \quad [\text{F23}]$$

$$\phi = \cos^{-1}\left[-\frac{\frac{r}{2} + \frac{a}{3}}{S}\right] \quad [\text{F24}]$$

Case 2a:  $r - \frac{a}{3} \geq 0$

For Case 2a, we set

$$\lambda_{11} = \sqrt{r - \frac{a}{3}} \quad [\text{F25}]$$

$$\lambda_{12} = -\sqrt{r - \frac{a}{3}} \quad [\text{F26}]$$

$$\lambda_{21} = \sqrt{S}e^{i\frac{\phi}{2}} = \sqrt{S}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right] \quad [\text{F27}]$$

$$\lambda_{22} = -\sqrt{S} e^{i\frac{\phi}{2}} = -\sqrt{S} \left[ \cos\frac{\phi}{2} + i \sin\frac{\phi}{2} \right] \quad [\text{F28}]$$

$$\lambda_{31} = \sqrt{S} e^{-i\frac{\phi}{2}} = \sqrt{S} \left[ \cos\frac{\phi}{2} - i \sin\frac{\phi}{2} \right] \quad [\text{F29}]$$

$$\lambda_{32} = -\sqrt{S} e^{-i\frac{\phi}{2}} = \sqrt{S} \left[ -\cos\frac{\phi}{2} + i \sin\frac{\phi}{2} \right] \quad [\text{F30}]$$

and then set

$$\lambda_1 = \sqrt{r - \frac{a}{3}} \quad [\text{F31}]$$

$$\lambda_2 = \sqrt{S} \cos\frac{\phi}{2} \quad [\text{F32}]$$

$$\lambda_3 = \sqrt{S} \sin\frac{\phi}{2} \quad [\text{F33}]$$

so that the mode functions are

$$w_1 = \cosh \lambda_1 \theta \quad [\text{F34}]$$

$$w_2 = \sinh \lambda_1 \theta \quad [\text{F35}]$$

$$w_3 = \cosh \lambda_2 \theta \cos \lambda_3 \theta \quad [\text{F36}]$$

$$w_4 = \cosh \lambda_2 \theta \sin \lambda_3 \theta \quad [\text{F37}]$$

$$w_5 = \sinh \lambda_2 \theta \cos \lambda_3 \theta \quad [\text{F38}]$$

$$w_6 = \sinh \lambda_2 \theta \sin \lambda_3 \theta \quad [\text{F39}]$$

Case 2b:  $r - \frac{a}{3} < 0$

For Case 2b, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3} - r} \quad [\text{F40}]$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3} - r} \quad [\text{F41}]$$

$$\lambda_{21} = \sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F27 bis}]$$

$$\lambda_{22} = -\sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F28 bis}]$$

$$\lambda_{31} = \sqrt{S} \left[ \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right] \quad [\text{F29 bis}]$$

$$\lambda_{32} = \sqrt{S} \left[ -\cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F30 bis}]$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3} - r} \quad [\text{F42}]$$

$$\lambda_2 = \sqrt{S} \cos \frac{\phi}{2} \quad [\text{F32 bis}]$$

$$\lambda_3 = \sqrt{S} \sin \frac{\phi}{2} \quad [\text{F33 bis}]$$

so that the mode functions are

$$W_1 = \cos \lambda_1 \theta \quad [F43]$$

$$W_2 = \sin \lambda_1 \theta \quad [F44]$$

$$W_3 = \cosh \lambda_2 \theta \cos \lambda_3 \theta \quad [F36 \text{ bis}]$$

$$W_4 = \cosh \lambda_2 \theta \sin \lambda_3 \theta \quad [F37 \text{ bis}]$$

$$W_5 = \sinh \lambda_2 \theta \cos \lambda_3 \theta \quad [F38 \text{ bis}]$$

$$W_6 = \sinh \lambda_2 \theta \sin \lambda_3 \theta \quad [F39 \text{ bis}]$$

Case 3:  $P = 0; Q < 0$

In Case 3, Equation E12 reduces to

$$y^3 + |Q| = 0 \quad [F45]$$

so that

$$y^3 = -|Q| \quad [F46]$$

Now let

$$r = |Q|^{1/3} > 0 \quad [F16 \text{ bis}]$$

Then

$$y_1 = -r \quad [F47]$$

$$y_2 = re^{i\frac{\pi}{3}} = \frac{r}{2} \left[ 1 + i\sqrt{3} \right] \quad [F48]$$

$$y_3 = re^{-i\frac{\pi}{3}} = \frac{r}{2} \left[ 1 - i\sqrt{3} \right] \quad [F49]$$

and Equations E45 yield

$$\lambda_I^2 = - \left[ r + \frac{a}{3} \right] \quad [F50]$$

$$\lambda_{II}^2 = \left[ \frac{r}{2} - \frac{a}{3} \right] + i \frac{r\sqrt{3}}{2} = Se^{i\phi} \quad [F51]$$

$$\lambda_{III}^2 = \left[ \frac{r}{2} - \frac{a}{3} \right] - i \frac{r\sqrt{3}}{2} = Se^{-i\phi} \quad [F52]$$

where

$$S^2 = \left[ \frac{r}{2} - \frac{a}{3} \right]^2 + \frac{3r^2}{4} = r^2 - \frac{ra}{3} + \frac{a^2}{9} \quad [F53]$$

$$\phi = \cos^{-1} \left[ \frac{\frac{r}{2} - \frac{a}{3}}{S} \right] \quad [F54]$$

We now set

$$\lambda_{11} = i\sqrt{r + \frac{a}{3}} \quad [F55]$$

$$\lambda_{12} = -i\sqrt{r + \frac{a}{3}} \quad [F56]$$

$$\lambda_{21} = \sqrt{S} e^{i\frac{\phi}{2}} = \sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [F27 \text{ bis}]$$

$$\lambda_{22} = -\sqrt{S} e^{i\frac{\phi}{2}} = -\sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [F28 \text{ bis}]$$

$$\lambda_{31} = \sqrt{S} e^{-i\frac{\phi}{2}} = \sqrt{S} \left[ \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right] \quad [F29 \text{ bis}]$$

$$\lambda_{32} = -\sqrt{S}e^{-i\frac{\phi}{2}} = \sqrt{S}\left[-\cos\frac{\phi}{2} + i \sin\frac{\phi}{2}\right] \quad [\text{F30 bis}]$$

and then set

$$\lambda_1 = \sqrt{r + \frac{a}{3}} \quad [\text{F57}]$$

$$\lambda_2 = \sqrt{S} \cos\frac{\phi}{2} \quad [\text{F32 bis}]$$

$$\lambda_3 = \sqrt{S} \sin\frac{\phi}{2} \quad [\text{F33 bis}]$$

so that the mode functions are

$$W_1 = \cos \lambda_1 \theta \quad [\text{F43 bis}]$$

$$W_2 = \sin \lambda_1 \theta \quad [\text{F44 bis}]$$

$$W_3 = \cosh \lambda_2 \theta \cos \lambda_3 \theta \quad [\text{F36 bis}]$$

$$W_4 = \cosh \lambda_2 \theta \sin \lambda_3 \theta \quad [\text{F37 bis}]$$

$$W_5 = \sinh \lambda_2 \theta \cos \lambda_3 \theta \quad [\text{F38 bis}]$$

$$W_6 = \sinh \lambda_2 \theta \sin \lambda_3 \theta \quad [\text{F39 bis}]$$

Case 4:  $P > 0$ ;  $Q = 0$

In Case 4, Equation E12 reduces to

$$y^3 - |P|y = y[y^2 - |P|] = 0 \quad [\text{F58}]$$

Now let

$$r = \sqrt{|P|} \quad [\text{F59}]$$

Then

$$y_1 = 0 \quad [\text{F60}]$$

$$y_2 = r \quad [\text{F61}]$$

$$y_3 = -r \quad [\text{F62}]$$

and Equations E45 yield

$$\lambda_I^2 = -\frac{a}{3} \quad [\text{F63}]$$

$$\lambda_{II}^2 = r - \frac{a}{3} \quad [\text{F64}]$$

$$\lambda_{III}^2 = -\left[r + \frac{a}{3}\right] \quad [\text{F65}]$$

Case 4a:  $r - \frac{a}{3} \geq 0$

For Case 4a, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3}} \quad [\text{F66}]$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3}} \quad [\text{F67}]$$

$$\lambda_{21} = \sqrt{r - \frac{a}{3}} \quad [\text{F68}]$$

$$\lambda_{22} = -\sqrt{r - \frac{a}{3}} \quad [\text{F69}]$$

$$\lambda_{31} = i\sqrt{r + \frac{a}{3}} \quad [\text{F70}]$$

$$\lambda_{32} = -i\sqrt{r + \frac{a}{3}} \quad [\text{F71}]$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3}} \quad [\text{F72}]$$

$$\lambda_2 = \sqrt{r - \frac{a}{3}} \quad [\text{F73}]$$

$$\lambda_3 = \sqrt{r + \frac{a}{3}} \quad [\text{F74}]$$

so that the mode functions are

$$w_1 = \cos \lambda_1 \theta \quad [\text{F43 bis}]$$

$$w_2 = \sin \lambda_1 \theta \quad [\text{F44 bis}]$$

$$w_3 = \cosh \lambda_2 \theta \quad [\text{F75}]$$

$$w_4 = \sinh \lambda_2 \theta \quad [\text{F76}]$$

$$w_5 = \cos \lambda_3 \theta \quad [\text{F77}]$$

$$w_6 = \sin \lambda_3 \theta \quad [\text{F78}]$$

Case 4b:  $r - \frac{a}{3} < 0$

For Case 4b, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3}} \quad [\text{F66 bis}]$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3}} \quad [\text{F67 bis}]$$

$$\lambda_{21} = i\sqrt{\frac{a}{3} - r} \quad [\text{F79}]$$



$$\lambda_{22} = -i\sqrt{\frac{a}{3} - r} \quad [\text{F80}]$$

$$\lambda_{31} = i\sqrt{r + \frac{a}{3}} \quad [\text{F70 bis}]$$

$$\lambda_{32} = -i\sqrt{r + \frac{a}{3}} \quad [\text{F71 bis}]$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3}} \quad [\text{F72 bis}]$$

$$\lambda_2 = \sqrt{\frac{a}{3} - r} \quad [\text{F81}]$$

$$\lambda_3 = \sqrt{r + \frac{a}{3}} \quad [\text{F74 bis}]$$

so that the mode functions are

$$w_1 = \cos \lambda_1 \theta \quad [\text{F43 bis}]$$

$$w_2 = \sin \lambda_1 \theta \quad [\text{F44 bis}]$$

$$w_3 = \cos \lambda_2 \theta \quad [\text{F82}]$$

$$w_4 = \sin \lambda_2 \theta \quad [\text{F83}]$$

$$w_5 = \cos \lambda_3 \theta \quad [\text{F77 bis}]$$

$$w_6 = \sin \lambda_3 \theta \quad [\text{F78 bis}]$$

Case 5:  $P < 0$ ;  $Q = 0$

In Case 5, Equation E12 reduces to

$$y^3 + |P|y = y[y^2 + |P|] = 0 \quad [\text{F84}]$$

Now let

$$r = \sqrt{|P|} \quad [\text{F59 bis}]$$

Then

$$y_1 = 0 \quad [\text{F60 bis}]$$

$$y_2 = ir \quad [\text{F85}]$$

$$y_3 = -ir \quad [\text{F86}]$$

and Equations E45 yield

$$\lambda_I^2 = -\frac{a}{3} \quad [\text{F63 bis}]$$

$$\lambda_{II}^2 = -\frac{a}{3} + ir = Se^{i\phi} \quad [\text{F87}]$$

$$\lambda_{III}^2 = -\frac{a}{3} - ir = Se^{-i\phi} \quad [\text{F88}]$$

where

$$S^2 = \frac{a^2}{9} + r^2 \quad [\text{F89}]$$

$$\phi = \cos^{-1} \left[ -\frac{a}{3S} \right] \quad [\text{F90}]$$

We now set

$$\lambda_{11} = i\sqrt{\frac{a}{3}} \quad [\text{F66 bis}]$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3}} \quad [\text{F67 bis}]$$

$$\lambda_{21} = \sqrt{S} e^{i\frac{\phi}{2}} = \sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F27 bis}]$$

$$\lambda_{22} = -\sqrt{S} e^{i\frac{\phi}{2}} = -\sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F28 bis}]$$

$$\lambda_{31} = \sqrt{S} e^{-i\frac{\phi}{2}} = \sqrt{S} \left[ \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right] \quad [\text{F29 bis}]$$

$$\lambda_{32} = -\sqrt{S} e^{-i\frac{\phi}{2}} = \sqrt{S} \left[ -\cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F30 bis}]$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3}} \quad [\text{F72 bis}]$$

$$\lambda_2 = \sqrt{S} \cos \frac{\phi}{2} \quad [\text{F32 bis}]$$

$$\lambda_3 = \sqrt{S} \sin \frac{\phi}{2} \quad [\text{F33 bis}]$$

so that the mode functions are

$$w_1 = \cos \lambda_1 \theta \quad [\text{F43 bis}]$$

$$w_2 = \sin \lambda_1 \theta \quad [\text{F44 bis}]$$

$$w_3 = \cosh \lambda_2 \theta \cos \lambda_3 \theta \quad [\text{F36 bis}]$$

$$w_4 = \cosh \lambda_2 \theta \sin \lambda_3 \theta \quad [\text{F37 bis}]$$

$$w_5 = \sinh \lambda_2 \theta \cos \lambda_3 \theta \quad [\text{F38 bis}]$$

$$w_6 = \sinh \lambda_2 \theta \sin \lambda_3 \theta \quad [\text{F39 bis}]$$

Case 6:  $P < 0$ ;  $Q \neq 0$

In Case 6,

$$\left[ \frac{p}{3} \right]^3 < 0 \quad [\text{F91}]$$

so that Equation E20 yields

$$\sqrt{R} = \sqrt{\left[\frac{Q}{2}\right]^2 - \left[\frac{P}{3}\right]^3} > \left|\frac{Q}{2}\right| \quad [\text{F92}]$$

and therefore Equations E21 and E22 yield

$$A^3 = \frac{Q}{2} + \sqrt{R} > 0 \quad [\text{F93}]$$

$$B^3 = \frac{Q}{2} - \sqrt{R} < 0 \quad [\text{F94}]$$

Now let

$$A = \left[\frac{Q}{2} + \sqrt{R}\right]^{1/3} \quad [\text{F95}]$$

$$B = \left[\frac{Q}{2} - \sqrt{R}\right]^{1/3} \quad [\text{F96}]$$

and then set

$$A_1 = A \quad [\text{F97}]$$

$$A_2 = Ae^{i\frac{2\pi}{3}} = \frac{A}{2}(-1 + i\sqrt{3}) \quad [\text{F98}]$$

$$A_3 = Ae^{-i\frac{2\pi}{3}} = -\frac{A}{2}(1 + i\sqrt{3}) \quad [\text{F99}]$$

$$B_1 = -B \quad [\text{F100}]$$

$$B_2 = Be^{i\frac{\pi}{3}} = \frac{B}{2}(1 + i\sqrt{3}) \quad [\text{F101}]$$

$$B_3 = Be^{-i\frac{\pi}{3}} = \frac{B}{2}(1 - i\sqrt{3}) \quad [\text{F102}]$$

Then Equation E32 yields

$$y_1 = A_1 + B_1 = A - B \quad [F103]$$

$$y_2 = A_2 + B_2 = \frac{1}{2} \left[ (-A + B) + i\sqrt{3}(A+B) \right] \quad [F104]$$

$$y_3 = A_3 + B_3 = \frac{1}{2} \left[ (-A + B) - i\sqrt{3}(A + B) \right] \quad [F105]$$

so that Equations E45 yield

$$\lambda_I^2 = (A - B) - \frac{a}{3} \quad [F106]$$

$$\lambda_{II}^2 = \left[ \frac{-A + B}{2} - \frac{a}{3} \right] + i\frac{\sqrt{3}}{2}(A + B) = Se^{i\phi} \quad [F107]$$

$$\lambda_{III}^2 = \left[ \frac{-A + B}{2} - \frac{a}{3} \right] - i\frac{\sqrt{3}}{2}(A + B) = Se^{-i\phi} \quad [F108]$$

where

$$\begin{aligned} S^2 &= \frac{A^2 - 2AB + B^2}{4} - \frac{(-A + B)a}{3} + \frac{a^2}{9} + \frac{3}{4}(A^2 + 2AB + B^2) \\ &= (A^2 + AB + B^2) + \frac{(A - B)a}{3} + \frac{a^2}{9} \end{aligned} \quad [F109]$$

$$\phi = \cos^{-1} \left[ -\frac{\frac{A - B}{2} + \frac{a}{3}}{S} \right] \quad [F110]$$

Case 6a:  $(A - B) - \frac{a}{3} \geq 0$

For Case 6a, we set

$$\lambda_{11} = \sqrt{(A - B) - \frac{a}{3}} \quad [F111]$$

$$\lambda_{12} = -\sqrt{(A - B) - \frac{a}{3}} \quad [\text{F112}]$$

$$\lambda_{21} = \sqrt{S} e^{i\frac{\phi}{2}} = \sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F27 bis}]$$

$$\lambda_{22} = -\sqrt{S} e^{i\frac{\phi}{2}} = -\sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F28 bis}]$$

$$\lambda_{31} = \sqrt{S} e^{-i\frac{\phi}{2}} = \sqrt{S} \left[ \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right] \quad [\text{F29 bis}]$$

$$\lambda_{32} = -\sqrt{S} e^{-i\frac{\phi}{2}} = \sqrt{S} \left[ -\cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F30 bis}]$$

and then set

$$\lambda_1 = \sqrt{(A - B) - \frac{a}{3}} \quad [\text{F113}]$$

$$\lambda_2 = \sqrt{S} \cos \frac{\phi}{2} \quad [\text{F32 bis}]$$

$$\lambda_3 = \sqrt{S} \sin \frac{\phi}{2} \quad [\text{F33 bis}]$$

so that the mode functions are

$$w_1 = \cosh \lambda_1 \theta \quad [\text{F34 bis}]$$

$$w_2 = \sinh \lambda_1 \theta \quad [\text{F35 bis}]$$

$$w_3 = \cosh \lambda_2 \theta \cos \lambda_3 \theta \quad [\text{F36 bis}]$$

$$w_4 = \cosh \lambda_2 \theta \sin \lambda_3 \theta \quad [\text{F37 bis}]$$

$$w_5 = \sinh \lambda_2 \theta \cos \lambda_3 \theta \quad [\text{F38 bis}]$$

$$w_6 = \sinh \lambda_2 \theta \sin \lambda_3 \theta \quad [\text{F39 bis}]$$

Case 6b:  $(A - B) - \frac{a}{3} < 0$

For Case 6b, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3} - (A - B)} \quad [\text{F114}]$$

$$\lambda_{12} = -i\sqrt{\frac{a}{3} - (A - B)} \quad [\text{F115}]$$

$$\lambda_{21} = \sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F27 bis}]$$

$$\lambda_{22} = -\sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F28 bis}]$$

$$\lambda_{31} = \sqrt{S} \left[ \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right] \quad [\text{F29 bis}]$$

$$\lambda_{32} = \sqrt{S} \left[ -\cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F30 bis}]$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3} - (A - B)} \quad [\text{F116}]$$

$$\lambda_2 = \sqrt{S} \cos \frac{\phi}{2} \quad [\text{F32 bis}]$$

$$\lambda_3 = \sqrt{S} \sin \frac{\phi}{2} \quad [\text{F33 bis}]$$

so that the mode functions are

$$W_1 = \cos \lambda_1 \theta \quad [\text{F43 bis}]$$

$$W_2 = \sin \lambda_1 \theta \quad [\text{F44 bis}]$$

$$W_3 = \cosh \lambda_2 \theta \cos \lambda_3 \theta \quad [\text{F36 bis}]$$

$$W_4 = \cosh \lambda_2 \theta \sin \lambda_3 \theta \quad [\text{F37 bis}]$$

$$W_5 = \sinh \lambda_2 \theta \cos \lambda_3 \theta$$

[F38 bis]

$$W_6 = \sinh \lambda_2 \theta \sin \lambda_3 \theta$$

[F39 bis]

$$\text{Case 7: } 0 < P \leq 3 \left[ \frac{Q}{2} \right]^{2/3}; Q > 0$$

In Case 7,

$$0 < \left[ \frac{P}{3} \right]^3 \leq \left[ \frac{Q}{2} \right]^2 \quad [\text{F117}]$$

so that Equation E20 yields

$$0 \leq \left[ \sqrt{R} = \sqrt{\left[ \frac{Q}{2} \right]^2 - \left[ \frac{P}{3} \right]^3} \right] < \left| \frac{Q}{2} \right| \quad [\text{F118}]$$

and therefore Equations E21 and E22 yield

$$A^3 = \frac{Q}{2} + \sqrt{R} > 0 \quad [\text{F119}]$$

$$B^3 = \frac{Q}{2} - \sqrt{R} > 0 \quad [\text{F120}]$$

Now let

$$A = \left[ \frac{Q}{2} + \sqrt{R} \right]^{1/3} \quad [\text{F121}]$$

$$B = \left[ \frac{Q}{2} - \sqrt{R} \right]^{1/3} \quad [\text{F122}]$$

and then set

$$A_1 = A \quad [\text{F97 bis}]$$

$$A_2 = A e^{i \frac{2\pi}{3}} = \frac{A}{2} (-1 + i\sqrt{3}) \quad [\text{F98 bis}]$$



$$A_3 = Ae^{-i\frac{2\pi}{3}} = -\frac{A}{2}(1 + i\sqrt{3}) \quad [\text{F99 bis}]$$

$$B_1 = B \quad [\text{F123}]$$

$$B_2 = Be^{i\frac{2\pi}{3}} = \frac{B}{2}(-1 + i\sqrt{3}) \quad [\text{F124}]$$

$$B_3 = Be^{-i\frac{2\pi}{3}} = -\frac{B}{2}(1 + i\sqrt{3}) \quad [\text{F125}]$$

Then Equation E32 yields

$$y_1 = A_1 + B_1 = A + B \quad [\text{F126}]$$

$$y_2 = A_2 + B_2 = \frac{A+B}{2}(-1 + i\sqrt{3}) \quad [\text{F127}]$$

$$y_3 = A_3 + B_3 = -\frac{A+B}{2}(1 + i\sqrt{3}) \quad [\text{F128}]$$

so that Equations E45 yield

$$\lambda_I^2 = (A + B) - \frac{a}{3} \quad [\text{F129}]$$

$$\lambda_{II}^2 = -\left[\frac{A+B}{2} + \frac{a}{3}\right] + i\frac{\sqrt{3}}{2}(A+B) = Se^{i\phi} \quad [\text{F130}]$$

$$\lambda_{III}^2 = -\left[\frac{A+B}{2} + \frac{a}{3}\right] - i\frac{\sqrt{3}}{2}(A+B) = Se^{-i\phi} \quad [\text{F131}]$$

where

$$\begin{aligned} S^2 &= \frac{(A+B)^2}{4} + \frac{(A+B)a}{3} + \frac{a^2}{9} + \frac{3}{4}(A+B)^2 \\ &= (A+B)^2 + \frac{(A+B)a}{3} + \frac{a^2}{9} \end{aligned} \quad [\text{F132}]$$

$$\phi = \cos^{-1} \left[ -\frac{\frac{A+B}{2} + \frac{a}{3}}{S} \right] \quad [\text{F133}]$$

Case 7a:  $(A+B) - \frac{a}{3} \geq 0$

For Case 7a, we set

$$\lambda_{11} = \sqrt{(A+B) - \frac{a}{3}} \quad [\text{F134}]$$

$$\lambda_{12} = -\sqrt{(A+B) - \frac{a}{3}} \quad [\text{F135}]$$

$$\lambda_{21} = \sqrt{S} e^{i\frac{\phi}{2}} = \sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F27 bis}]$$

$$\lambda_{22} = -\sqrt{S} e^{i\frac{\phi}{2}} = -\sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F28 bis}]$$

$$\lambda_{31} = \sqrt{S} e^{-i\frac{\phi}{2}} = \sqrt{S} \left[ \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right] \quad [\text{F29 bis}]$$

$$\lambda_{32} = -\sqrt{S} e^{-i\frac{\phi}{2}} = \sqrt{S} \left[ -\cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [\text{F30 bis}]$$

and then set

$$\lambda_1 = \sqrt{(A+B) - \frac{a}{3}} \quad [\text{F136}]$$

$$\lambda_2 = \sqrt{S} \cos \frac{\phi}{2} \quad [\text{F32 bis}]$$

$$\lambda_3 = \sqrt{S} \sin \frac{\phi}{2} \quad [\text{F33 bis}]$$

so that the mode functions are

$$\begin{aligned} W_1 &= \cosh \lambda_1 \theta & [F34 \text{ bis}] \\ W_2 &= \sinh \lambda_1 \theta & [F35 \text{ bis}] \\ W_3 &= \cosh \lambda_2 \theta \cos \lambda_3 \theta & [F36 \text{ bis}] \\ W_4 &= \cosh \lambda_2 \theta \sin \lambda_3 \theta & [F37 \text{ bis}] \\ W_5 &= \sinh \lambda_2 \theta \cos \lambda_3 \theta & [F38 \text{ bis}] \\ W_6 &= \sinh \lambda_2 \theta \sin \lambda_3 \theta & [F39 \text{ bis}] \end{aligned}$$

Case 7b:  $(A + B) - \frac{a}{3} < 0$

For Case 7b, we set

$$\lambda_{11} = i \sqrt{\frac{a}{3} - (A + B)} \quad [F137]$$

$$\lambda_{12} = -i \sqrt{\frac{a}{3} - (A + B)} \quad [F138]$$

$$\lambda_{21} = \sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [F27 \text{ bis}]$$

$$\lambda_{22} = -\sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [F28 \text{ bis}]$$

$$\lambda_{31} = \sqrt{S} \left[ \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right] \quad [F29 \text{ bis}]$$

$$\lambda_{32} = \sqrt{S} \left[ -\cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [F30 \text{ bis}]$$

and then set

$$\lambda_1 = \sqrt{\frac{a}{3} - (A + B)} \quad [F139]$$

$$\lambda_2 = \sqrt{S} \cos \frac{\phi}{2} \quad [F32 \text{ bis}]$$

$$\lambda_3 = \sqrt{S} \sin \frac{\phi}{2}$$

[F33 bis]

so that the mode functions are

$$W_1 = \cos \lambda_1 \theta$$

[F43 bis]

$$W_2 = \sin \lambda_1 \theta$$

[F44 bis]

$$W_3 = \cosh \lambda_2 \theta \cos \lambda_3 \theta$$

[F36 bis]

$$W_4 = \cosh \lambda_2 \theta \sin \lambda_3 \theta$$

[F37 bis]

$$W_5 = \sinh \lambda_2 \theta \cos \lambda_3 \theta$$

[F38 bis]

$$W_6 = \sinh \lambda_2 \theta \sin \lambda_3 \theta$$

[F39 bis]

Case 8:  $0 < P \leq 3 \left[ \frac{Q}{2} \right]^{2/3}; Q < 0$

In Case 8,

$$0 < \left[ \frac{P}{3} \right]^3 \leq \left[ \frac{Q}{2} \right]^2$$

[F117 bis]

so that Equation E20 yields

$$0 \leq \left[ \sqrt{R} = \sqrt{\left[ \frac{Q}{2} \right]^2 - \left[ \frac{P}{3} \right]^3} \right] < \left| \frac{Q}{2} \right|$$

[F118 bis]

and therefore Equations E21 and E22 yield

$$A^3 = \frac{Q}{2} + \sqrt{R} < 0$$

[F140]

$$B^3 = \frac{Q}{2} - \sqrt{R} < 0$$

[F141]

Now let

$$A = \left| \frac{Q}{2} + \sqrt{R} \right|^{1/3}$$

[F142]

$$B = \left| \frac{Q}{2} - \sqrt{R} \right|^{1/3}$$

[F143]

and then set

$$A_1 = -A \quad [F144]$$

$$A_2 = Ae^{i\frac{\pi}{3}} = \frac{A}{2}(1 + i\sqrt{3}) \quad [F145]$$

$$A_3 = Ae^{-i\frac{\pi}{3}} = \frac{A}{2}(1 - i\sqrt{3}) \quad [F146]$$

$$B_1 = -B \quad [F147]$$

$$B_2 = Be^{i\frac{\pi}{3}} = \frac{B}{2}(1 + i\sqrt{3}) \quad [F148]$$

$$B_3 = Be^{-i\frac{\pi}{3}} = \frac{B}{2}(1 - i\sqrt{3}) \quad [F149]$$

Then Equation E32 yields

$$y_1 = A_1 + B_1 = -(A + B) \quad [F150]$$

$$y_2 = A_2 + B_2 = \frac{A + B}{2}(1 + i\sqrt{3}) \quad [F151]$$

$$y_3 = A_3 + B_3 = \frac{A + B}{2}(1 - i\sqrt{3}) \quad [F152]$$

so that Equations E45 yield

$$\lambda_I^2 = -\left[(A + B) + \frac{a}{3}\right] \quad [F153]$$

$$\lambda_{II}^2 = \left[\frac{A + B}{2} - \frac{a}{3}\right] + i\frac{\sqrt{3}}{2}(A + B) = Se^{i\phi} \quad [F154]$$

$$\lambda_{III}^2 = \left[\frac{A + B}{2} - \frac{a}{3}\right] - i\frac{\sqrt{3}}{2}(A + B) = Se^{-i\phi} \quad [F155]$$

where

$$\begin{aligned} S^2 &= \frac{(A+B)^2}{4} - \frac{(A+B)a}{3} + \frac{a^2}{9} + \frac{3}{4}(A+B)^2 \\ &= (A+B)^2 - \frac{(A+B)a}{3} + \frac{a^2}{9} \end{aligned} \quad [F156]$$

$$\phi = \cos^{-1} \left[ \frac{\frac{A+B}{2} - \frac{a}{3}}{S} \right] \quad [F157]$$

We then set

$$\lambda_{11} = i \sqrt{(A+B) + \frac{a}{3}} \quad [F158]$$

$$\lambda_{12} = -i \sqrt{(A+B) + \frac{a}{3}} \quad [F159]$$

$$\lambda_{21} = \sqrt{S} e^{i \frac{\phi}{2}} = \sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [F27 \text{ bis}]$$

$$\lambda_{22} = -\sqrt{S} e^{i \frac{\phi}{2}} = -\sqrt{S} \left[ \cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [F28 \text{ bis}]$$

$$\lambda_{31} = \sqrt{S} e^{-i \frac{\phi}{2}} = \sqrt{S} \left[ \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \right] \quad [F29 \text{ bis}]$$

$$\lambda_{32} = -\sqrt{S} e^{-i \frac{\phi}{2}} = \sqrt{S} \left[ -\cos \frac{\phi}{2} + i \sin \frac{\phi}{2} \right] \quad [F30 \text{ bis}]$$

and then set

$$\lambda_1 = \sqrt{(A+B) + \frac{a}{3}} \quad [F160]$$

$$\lambda_2 = \sqrt{S} \cos \frac{\phi}{2} \quad [F32 \text{ bis}]$$

$$\lambda_3 = \sqrt{S} \sin \frac{\phi}{2}$$

[F33 bis]

so that the mode functions are

$$w_1 = \cos \lambda_1 \theta$$

[F43 bis]

$$w_2 = \sin \lambda_1 \theta$$

[F44 bis]

$$w_3 = \cosh \lambda_2 \theta \cos \lambda_3 \theta$$

[F36 bis]

$$w_4 = \cosh \lambda_2 \theta \sin \lambda_3 \theta$$

[F37 bis]

$$w_5 = \sinh \lambda_2 \theta \cos \lambda_3 \theta$$

[F38 bis]

$$w_6 = \sinh \lambda_2 \theta \sin \lambda_3 \theta$$

[F39 bis]

Case 9:  $P > 3 \left[ \frac{Q}{2} \right]^{2/3}$ ;  $Q \neq 0$

In Case 9,

$$\left[ \frac{P}{3} \right]^3 > \left[ \frac{Q}{2} \right]^2$$

[F161]

so that Equation E20 yields

$$R = \left[ \frac{Q}{2} \right]^2 - \left[ \frac{P}{3} \right]^3 < 0$$

[F162]

We therefore define

$$\omega = \frac{1}{3} \cos^{-1} \left[ \frac{\left[ \frac{Q}{2} \right]}{\left[ \frac{P}{3} \right]^{3/2}} \right] \quad (0 \leq \omega \leq \frac{\pi}{3})$$

[F163]

and then set

$$\omega_1 = \omega$$

[E39 bis]

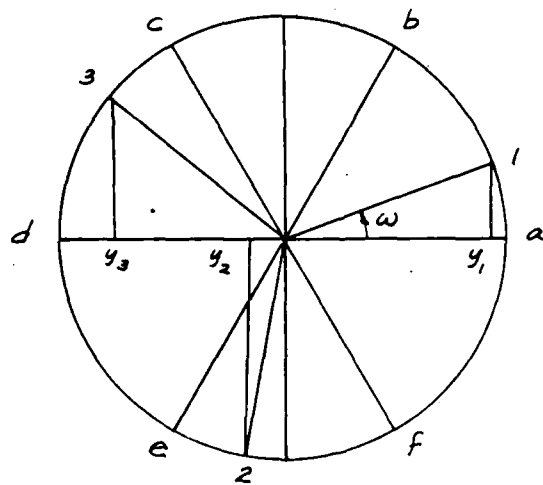
$$\omega_2 = \omega - \frac{2\pi}{3}$$

[E40 bis]

[E41 bis]

$$y_1 = 2\sqrt{\frac{p}{3}} \cos \omega_1 \quad [\text{F164}]$$

$$y_3 = 2\sqrt{\frac{P}{3}} \cos \omega_3 \quad [\text{F166}]$$

$$0 \leq \omega \leq \frac{\pi}{3} \quad [\text{F167}]$$
$$y_1 \geq y_2 \geq y_3 \quad [F168]$$




We then set

$$\lambda_I^2 = y_1 - \frac{a}{3} \quad [F169]$$

$$\lambda_{II}^2 = y_2 - \frac{a}{3} \quad [F170]$$

$$\lambda_{III}^2 = y_3 - \frac{a}{3} \quad [F171]$$

These cases are possible, as shown below.

		$\lambda_I^2$	
		$\geq 0$	$< 0$
$\lambda_{II}^2$	$\geq 0$	9a	<del>9c</del>
	$< 0$	9b	9c

Case 9a:  $\lambda_I^2 \geq 0$ ;  $\lambda_{II}^2 \geq 0$

In Case 9a, we set

$$\lambda_{11} = \sqrt{y_1 - \frac{a}{3}} \quad [F172]$$

$$\lambda_{12} = -\sqrt{y_1 - \frac{a}{3}} \quad [F173]$$

$$\lambda_{21} = \sqrt{y_2 - \frac{a}{3}} \quad [F174]$$

$$\lambda_{22} = -\sqrt{y_2 - \frac{a}{3}} \quad [F175]$$

$$\lambda_{31} = i\sqrt{\frac{a}{3} - y_3} \quad [F176]$$

$$\lambda_{32} = -i\sqrt{\frac{a}{3} - y_3} \quad [F177]$$

and then set

$$\lambda_1 = \sqrt{y_1 - \frac{a}{3}} \quad [\text{F178}]$$

$$\lambda_2 = \sqrt{y_2 - \frac{a}{3}} \quad [\text{F179}]$$

$$\lambda_3 = \sqrt{\frac{a}{3} - y_3} \quad [\text{F180}]$$

so that the mode functions are

$$w_1 = \cosh \lambda_1 \theta \quad [\text{F34 bis}]$$

$$w_2 = \sinh \lambda_1 \theta \quad [\text{F35 bis}]$$

$$w_3 = \cosh \lambda_2 \theta \quad [\text{F75 bis}]$$

$$w_4 = \sinh \lambda_2 \theta \quad [\text{F76 bis}]$$

$$w_5 = \cos \lambda_3 \theta \quad [\text{F77 bis}]$$

$$w_6 = \sin \lambda_3 \theta \quad [\text{F78 bis}]$$

Case 9b:  $\lambda_I^2 \geq 0$ ;  $\lambda_{II}^2 < 0$

In Case 9b, we set

$$\lambda_{11} = \sqrt{y_1 - \frac{a}{3}} \quad [\text{F172 bis}]$$

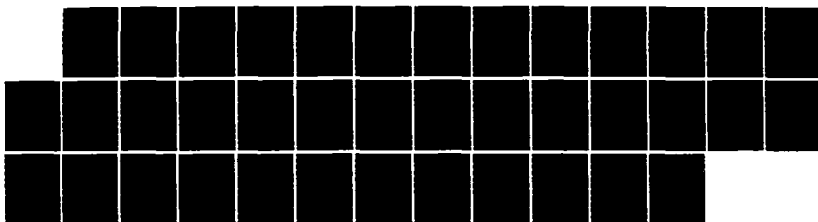
$$\lambda_{12} = -\sqrt{y_1 - \frac{a}{3}} \quad [\text{F173 bis}]$$

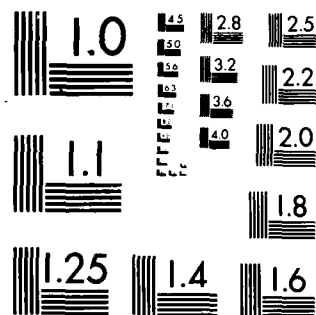
$$\lambda_{21} = i\sqrt{\frac{a}{3} - y_2} \quad [\text{F181}]$$

$$\lambda_{22} = -i\sqrt{\frac{a}{3} - y_2} \quad [\text{F182}]$$

$$\lambda_{31} = i\sqrt{\frac{a}{3} - y_3} \quad [\text{F176 bis}]$$

AD-A174 503 HIGH FREQUENCY ANALYSIS OF CIRCULAR ARCHES(U) APPLIED 2/2  
RESEARCH ASSOCIATES INC ALBUQUERQUE NM  
D H MERKLE ET AL. SEP 86 3956 AFML-TR-86-22  
UNCLASSIFIED F29601-85-C-0029 F/G 13/13 NL





MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

$$\lambda_{32} = -i\sqrt{\frac{a}{3} - y_3}$$

[F177 bis]

and then set

$$\lambda_1 = \sqrt{y_1 - \frac{a}{3}}$$

[F178 bis]

$$\lambda_2 = \sqrt{\frac{a}{3} - y_2}$$

[F183]

$$\lambda_3 = \sqrt{\frac{a}{3} - y_3}$$

[F180 bis]

so that the mode functions are

$$w_1 = \cosh \lambda_1 \theta$$

[F34 bis]

$$w_2 = \sinh \lambda_1 \theta$$

[F35 bis]

$$w_3 = \cos \lambda_2 \theta$$

[F82 bis]

$$w_4 = \sin \lambda_2 \theta$$

[F83 bis]

$$w_5 = \cos \lambda_3 \theta$$

[F77 bis]

$$w_6 = \sin \lambda_3 \theta$$

[F78 bis]

Case 9c:  $\lambda_I^2 < 0$

In Case 9c, we set

$$\lambda_{11} = i\sqrt{\frac{a}{3} - y_1}$$

[F184]

$$\lambda_{12} = -i\sqrt{\frac{a}{3} - y_1}$$

[F185]

$$\lambda_{21} = i\sqrt{\frac{a}{3} - y_2}$$

[F181 bis]

$$\lambda_{22} = -i\sqrt{\frac{a}{3} - y_2}$$

[F182 bis]

$$\lambda_{31} = i\sqrt{\frac{a}{3} - y_3}$$

[F176 bis]

$$\lambda_{32} = -i\sqrt{\frac{a}{3} - y_3}$$

[F177 bis]

and then set

$$\lambda_1 = \sqrt{\frac{a}{3} - y_1}$$

[F186]

$$\lambda_2 = \sqrt{\frac{a}{3} - y_2}$$

[F183 bis]

$$\lambda_3 = \sqrt{\frac{a}{3} - y_3}$$

[F180 bis]

so that the mode functions are

$$w_1 = \cos \lambda_1 \theta$$

[F43 bis]

$$w_2 = \sin \lambda_1 \theta$$

[F44 bis]

$$w_3 = \cos \lambda_2 \theta$$

[F82 bis]

$$w_4 = \sin \lambda_2 \theta$$

[F83 bis]

$$w_5 = \cos \lambda_3 \theta$$

[F77 bis]

$$w_6 = \sin \lambda_3 \theta$$

[F78 bis]

#### Cases 1-9 (Summary)

For convenience, the above cases are summarized below.

<u>Case</u>	<u>Criteria</u>
1	$P = Q = 0$
2	$P = 0; Q > 0$
2a	$r - a/3 \geq 0$
2b	$r - a/3 < 0$
3	$P = 0; Q < 0$

<u>Case</u>	<u>Criteria</u>
4	$P > 0; Q = 0$
4a	$r - a/3 \geq 0$
4b	$r - a/3 < 0$
5	$P < 0; Q = 0$
6	$P < 0; Q \neq 0$
6a	$(A - B) - a/3 \geq 0$
6b	$(A - B) - a/3 < 0$
7	$0 < P \leq 3 \left[ \frac{Q}{2} \right]^{2/3}; Q > 0$
7a	$(A + B) - a/3 \geq 0$
7b	$(A + B) - a/3 < 0$
8	$0 < P \leq 3 \left[ \frac{Q}{2} \right]^{2/3}; Q < 0$
9	$P > 3 \left[ \frac{Q}{2} \right]^{2/3}; Q \neq 0$
9a	$\lambda_I^2 \geq 0; \lambda_{II}^2 \geq 0$
9b	$\lambda_I^2 \geq 0; \lambda_{II}^2 < 0$
9c	$\lambda_I^2 < 0$

## APPENDIX G

### SYMMETRY

Any function,  $f(x)$ , can be expressed as the sum of a symmetric (even) function,  $f_E(x)$ , and an antisymmetric (odd) function,  $f_O(x)$ ,

$$f(x) = f_E(x) + f_O(x) \quad [G1]$$

where

$$f_E(x) = \frac{f(x) + f(-x)}{2} \quad [G2]$$

$$f_O(x) = \frac{f(x) - f(-x)}{2} \quad [G3]$$

From Equations G2 and G3 come the definitions of an even and an odd function.

$$f_E(-x) = f_E(x) \quad [G4]$$

$$f_O(-x) = -f_O(x) \quad [G5]$$

From the definition of a derivative, and using the notation

$$L = \lim_{\Delta x \rightarrow 0} \quad [G6]$$

we obtain

$$\begin{aligned} f_E'(-x) &= L \left[ \frac{f_E(-x) - f_E(-x - \Delta x)}{\Delta x} \right] \\ &= L \left[ \frac{f_E(x) - f_E(x + \Delta x)}{\Delta x} \right] = -f_E'(x) \end{aligned} \quad [G7]$$

$$\begin{aligned} f_O'(-x) &= L \left[ \frac{f_O(-x) - f_O(-x - \Delta x)}{\Delta x} \right] \\ &= L \left[ \frac{-f_O(x) + f_O(x + \Delta x)}{\Delta x} \right] = f_O'(x) \end{aligned} \quad [G8]$$



It follows that the even and odd derivatives of even and odd functions are even or odd according to the table below.

		DERIVATIVE	
		EVEN	ODD
FUNCTION	EVEN	EVEN	ODD
	ODD	ODD	EVEN

## APPENDIX H

### MODE SHAPE DETERMINATION

Equations B5, C3, and C4 express the centroidal translation and rotation of a plane cross section of a circular arch vibrating in a single mode in separable form,

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ r\psi \end{Bmatrix} = T \{U\} = T \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \quad [H1]$$

where, referring to Figure A1

$u_1$  = centroidal tangential displacement, in the direction of increasing  $\theta$

$u_2$  = centroidal inward radial displacement

$\psi$  = clockwise rotation

$r$  = arch cross-section radius of gyration, defined in Equation A62

$T$  = modal amplitude, a function of time only

$X, Y, Z$  = mode shapes, functions of the arch angle,  $\theta$ , only

Equation H1 is the general definition of a structural vibratory mode, i.e., a condition in which the relative displacements of all points on the structure remain constant, and the absolute displacements are proportional to a single scalar function of time. The configuration defined is that of the arch cross-section centroidal axis, not its neutral axis, because the centroidal axis location is independent of cross-section rotation.

When the arch undergoes free vibration, Equation C15 shows that the modal amplitude,  $T$ , is harmonic with angular frequency  $p$ .

$$\ddot{T} + p^2 T = 0 \quad [C15 \text{ bis}]$$

Appendix F shows that the three mode shapes,  $X$  (for centroidal tangential displacement),  $Y$  (for centroidal radial displacement), and  $Z$  (for plane cross-section rotation) can each be expressed as a linear combination of six mode functions,

$$\{U\} = \underline{C} \{W\} \quad [H2]$$

where

$\{U\} = 3 \times 1$  column matrix of mode shapes

$\{W\} = 6 \times 1$  column matrix of mode functions

$\underline{C} = 3 \times 6$  rectangular matrix of mode shape coefficients

Each of the six mode functions,  $W_i$  ( $i=1,6$ ), satisfies Equation E2, which is a homogeneous, linear, ordinary differential equation with constant coefficients, of order six. Equation E2 is derived from Equations C18, which are three independent, homogeneous, linear, ordinary differential equations with constant coefficients, of order two. Equations C18 are the fundamental equations describing arch free vibration, and Equation E2 was derived from them for the sake of convenience, to uncouple the system (cf. Equations C21-C23). However, the price of uncoupling the system is that Equation H2 is too general. Unless restrictions are placed on the mode shape coefficients, Equations C18 will not be satisfied. To determine what restrictions must be placed on the mode shape coefficients, we substitute Equation H2 into Equation C18 and obtain

$$(\underline{L} + \sigma^2 \underline{I}) \{U\} = \underline{C}(\underline{L} + \sigma^2 \underline{I}) \{W\} = \{0\} \quad [H3]$$

where

$$\sigma = \frac{p}{\left[ \frac{C}{R} \right]} \quad [C17 \text{ bis}]$$

$$c^2 = \frac{E}{\rho} \quad [B16 \text{ bis}]$$

and

$E$  = Young's elastic modulus

$\rho$  = mass density

$R$  = arch centroidal radius

The six mode functions,  $W_i$ , are a closed set, in the sense that differentiation of any one mode function yields a linear combination of the other five. Equations H3 are thus a set of three homogeneous linear equations in

the six mode functions. But the six mode functions are linearly independent, so the coefficient matrix in the resulting equations must be the null matrix (all elements zero), i.e.,

$$\underline{C}(\underline{L} + \sigma^2 \underline{I})(\underline{W}) = \underline{C} \underline{E}(\underline{W}) = \{0\} \quad [H4]$$

where

$$\underline{C} \underline{E} = \underline{0} \quad [H5]$$

However, since Equation E2 already represents all three of Equations C18 taken together, only two rows of Equation H4 are linearly independent. Therefore only 12 of the 18 scalar equations represented by Equation H5 are independent. Equation H5 therefore yields expressions for 12 of the 18 mode shape coefficients as linear combinations of the remaining 6.

The relative values of the six remaining mode shape coefficients are determined from the six arch boundary conditions, which take the form

$$\underline{B}(\underline{C}) = \{0\} \quad [H6]$$

where

$\underline{C}$  = 6 x 1 column matrix of remaining mode shape coefficients

$\underline{B}$  = 6 x 6 square matrix

The elements of the  $\underline{B}$  matrix are linear combinations of the six mode functions, defined in Appendix F, evaluated at an arch boundary, the precise forms of which depend on the frequency parameter,  $\sigma$ . For a nontrivial solution to Equation [H6] to exist, the determinant of the  $\underline{B}$  matrix must vanish.

$$|\underline{B}| = 0 \quad [H7]$$

When this happens, the frequency parameter,  $\sigma$ , corresponds to a natural or modal frequency.

What makes the arch vibration problem more complicated than some other structural vibration problems is the fact that the precise form of the

transcendental frequency equation generated by Equation H7 depends on the frequency parameter,  $\sigma$ . Thus in a trial and error solution to find the natural frequencies and mode shapes of a circular arch, the form of the frequency equation varies with the trial frequency parameter,  $\sigma$ .

# APPENDIX I

## ORTHOGONALITY RELATIONS

The basic equations for the mode shapes of a circular arch are

$$(\underline{L} + \sigma^2 \underline{I}) \{U\} = \{0\} \quad [C18 \text{ bis}]$$

where

$$\underline{L} = \begin{bmatrix} AD^2 - B & -(A+B)D & -ED^2 + BF \\ (A+B)D & BD^2 - A & -(BF+E)D \\ -ED^2 + BF & (BF+E)D & CD^2 - BF^2 \end{bmatrix} \quad [D8 \text{ bis}]$$

and

$$D = \frac{d}{d\theta} \quad [A60 \text{ bis}]$$

$$A = C_1 \quad [D3 \text{ bis}]$$

$$B = \frac{k'}{2(1+\nu)} \quad [D4 \text{ bis}]$$

$$C = C_3 \quad [D5 \text{ bis}]$$

$$E = 2\sqrt{3}C_2 \quad [D6 \text{ bis}]$$

$$F = \frac{R}{r} \quad [D7 \text{ bis}]$$

The constants  $C_1$ ,  $C_2$ , and  $C_3$  are functions of the ratio of arch depth to radius, as shown in the table following Equation A58.

To investigate the modal orthogonality relations we evaluate Equation C18 for modes  $i$  and  $j$ .

$$(\underline{L} + \sigma_i^2 \underline{I}) \{U\}_i = \{0\} \quad [I1]$$

$$(\underline{L} + \sigma_j^2 \underline{I}) \{U\}_j = \{0\} \quad [I2]$$

Premultiplying Equation I1 by  $\{U\}_j^T$ , and premultiplying Equation I2 by  $\{U\}_i^T$  yields

$$\{U\}_j^T \underline{L} \{U\}_i + \sigma_i^2 \{U\}_j^T \{U\}_i = 0 \quad [I3]$$

$$\{U\}_i^T \underline{L} \{U\}_j + \sigma_j^2 \{U\}_i^T \{U\}_j = 0 \quad [I4]$$

Subtracting Equation I4 from Equation I3 yields

$$\{U\}_j^T \underline{L} \{U\}_i - \{U\}_i^T \underline{L} \{U\}_j + [\sigma_i^2 - \sigma_j^2] \{U\}_i^T \{U\}_j = 0$$

or

$$[\sigma_i^2 - \sigma_j^2] \{U\}_i^T \{U\}_j = \{U\}_i^T \underline{L} \{U\}_j - \{U\}_j^T \underline{L} \{U\}_i \quad [I5]$$

It remains to be shown that the RHS of Equation I5 is the derivative of a function which has equal values at  $\theta = 0$  and  $\theta = \Omega$ , so that

$$\{U\}_i^T \underline{L} \{U\}_j - \{U\}_j^T \underline{L} \{U\}_i = B'_{ij} \quad [I6]$$

and

$$\left[ B_{ij}(\theta) \right]_0^\Omega = 0 \quad [I7]$$

If Equations I6 and I7 are satisfied, then integration of Equation I5 on  $\theta$  between 0 and  $\Omega$  yields

$$[\sigma_i^2 - \sigma_j^2] \int_0^\Omega \{U\}_i^T \{U\}_j d\theta = \int_0^\Omega B'_{ij} d\theta = \left[ B_{ij}(\theta) \right]_0^\Omega = 0 \quad [I8]$$

When  $i \neq j$ , so that  $\sigma_i^2 \neq \sigma_j^2$ , Equation I8 yields

$$\int_0^\Omega \{U\}_i^T \{U\}_j d\theta = 0 \quad (i \neq j) \quad [I9]$$

Thus the  $i^{\text{th}}$  and  $j^{\text{th}}$  modes are orthogonal over the closed interval  $0 \leq \theta \leq \Omega$ , provided Equations I6 and I7 are satisfied.

To prove Equation I6 we expand its LHS using Equation D8, and obtain

$$\begin{aligned}
B'_{ij} &= A \left[ X_i X_j'' - X_j X_i'' \right] - (A + B) \left[ X_i Y_j' - X_j Y_i' \right] - E \left[ X_i Z_j'' - X_j Z_i'' \right] \\
&\quad + (A + B) \left[ Y_i X_j' - Y_j X_i' \right] + B \left[ Y_i Y_j'' - Y_j Y_i'' \right] - (BF + E) \left[ Y_i Z_j' - Y_j Z_i' \right] \\
&\quad - E \left[ Z_i X_j'' - Z_j X_i'' \right] + (BF + E) \left[ Z_i Y_j' - Z_j Y_i' \right] + C \left[ Z_i Z_j'' - Z_j Z_i'' \right] \\
&= A \left[ X_i X_j'' - X_j X_i'' \right] + B \left[ Y_i Y_j'' - Y_j Y_i'' \right] + C \left[ Z_i Z_j'' - Z_j Z_i'' \right] \\
&\quad - (A + B) \left[ \left[ X_i' Y_j + X_i Y_j' \right] - \left[ X_j' Y_i + X_j Y_i' \right] \right] \\
&\quad - (BF + E) \left[ \left[ Y_i' Z_j + Y_i Z_j' \right] - \left[ Y_j' Z_i + Y_j Z_i' \right] \right] \\
&\quad + E \left[ \left[ Z_i'' X_j - Z_i X_j'' \right] - \left[ Z_j'' X_i - Z_j X_i'' \right] \right] \tag{I10}
\end{aligned}$$

Now

$$\left[ X_i X_j' \right]' - \left[ X_i' X_j \right]' = X_i' X_j' + X_i X_j'' - X_i'' X_j - X_i' X_j' = X_i X_j'' - X_j X_i'' \tag{I11}$$

and

$$\left[ Z_i' X_j \right]' - \left[ Z_i X_j' \right]' = Z_i'' X_j + Z_i' X_j' - Z_i' X_j' - Z_i X_j'' = Z_i'' X_j - Z_j X_i'' \tag{I12}$$

Therefore, Equation I10 can be written in the form

$$\begin{aligned}
B'_{ij} &= A \left[ \left[ X_i X_j' \right]' - \left[ X_i' X_j \right]' \right] + B \left[ \left[ Y_i Y_j' \right]' - \left[ Y_i' Y_j \right]' \right] \\
&\quad + C \left[ \left[ Z_i Z_j' \right]' - \left[ Z_i' Z_j \right]' \right] - (A + B) \left[ \left[ X_i Y_j \right]' - \left[ X_j Y_i \right]' \right]
\end{aligned}$$



$$\begin{aligned}
& - (BF + E) \left[ \left[ Y_i Z_j \right]' - \left[ Y_j Z_i \right]' \right] \\
& + E \left[ \left[ Z_i X_j \right]' - \left[ Z_i X_j \right]' - \left[ Z_j X_i \right]' + \left[ Z_j X_i \right]' \right]
\end{aligned} \tag{I13}$$

Each term on the RHS of Equation I13 is a derivative, so Equation I6 is proven.

To prove Equation I7 we expand Equations B8 and B9.

$$\{F\} = E^* A^* \underline{S} \{\delta\} \tag{B8 bis}$$

$$\{\delta\} = \frac{1}{R} (D\underline{I} - \underline{G}^T) \{u\} \tag{B9 bis}$$

where, to avoid confusion with symbols already used in this appendix

$E^*$  = Young's elastic modulus

$A^*$  = arch cross-sectional area

Substituting Equations B9 and C3 into Equation B8 yields

$$\begin{aligned}
\{F\} &= \frac{E^* A^*}{R} \underline{S} (D\underline{I} - \underline{G}^T) \{u\} = \frac{E^* A^*}{R} (D\underline{S} - \underline{S}\underline{G}^T) \{u\} \\
&= \frac{E^* A^*}{R} T (D\underline{S} - \underline{S}\underline{G}^T) \{u\}
\end{aligned} \tag{I14}$$

Now

$$\underline{S} = \begin{bmatrix} C_1 & 0 & -2\sqrt{3}C_2 \\ 0 & \frac{k}{2(1+\nu)} & 0 \\ -2\sqrt{3}C_2 & 0 & C_3 \end{bmatrix} \tag{B2, B22 bis}$$

$$\underline{SG}^T = \begin{bmatrix} 0 & C_1 & 0 \\ -\frac{k'}{2(1+\nu)} & 0 & \frac{k'R}{2r(1+\nu)} \\ 0 & -2\sqrt{3}C_2 & 0 \end{bmatrix} \quad [\text{B13 bis}]$$

so that

$$\begin{aligned} \underline{DS} - \underline{SG}^T &= \begin{bmatrix} C_1 D & -C_1 & -2\sqrt{3}C_2 D \\ \frac{k'}{2(1+\nu)} & \frac{k'}{2(1+\nu)} D & -\frac{k'R}{2r(1+\nu)} \\ -2\sqrt{3}C_2 D & 2\sqrt{3}C_2 & C_3 D \end{bmatrix} \\ &= \begin{bmatrix} AD & -A & -ED \\ B & BD & -BF \\ -ED & E & CD \end{bmatrix} \end{aligned} \quad [\text{I15}]$$

Now if

$$\{F\} = \begin{bmatrix} F_1 \\ F_2 \\ M \\ r \end{bmatrix} = \{N\}^T = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}^T \quad [\text{I16}]$$

then substituting Equations I15, I16, and C3 into Equation I14 yields

$$N_1 = \frac{E^* A^*}{R} [A(X' - Y) - EZ'] \quad [\text{I17}]$$

$$N_2 = \frac{E^* A^*}{R} [B(X + Y' - FZ)] \quad [\text{I18}]$$

$$N_3 = \frac{E^* A^*}{R} [E(Y - X') + CZ'] \quad [\text{I19}]$$

By rearranging terms, Equation I13 can be written in the form

$$B'_{ij} = \left[ X_i [A[X'_j - Y_j] - EZ'_j] \right]' - \left[ X_j [A[X'_i - Y_i] - EZ'_i] \right]'$$

$$\begin{aligned}
& + \left[ Y_i B \left[ X_j + Y_j' - FZ_j \right] \right]' - \left[ Y_j B \left[ X_i + Y_i' - FZ_i \right] \right]' \\
& + \left[ Z_i \left[ E \left[ Y_j - X_j' \right] + CZ_j' \right] \right]' - \left[ Z_j \left[ E \left[ Y_i - X_i' \right] + CZ_i' \right] \right]' \quad [I20]
\end{aligned}$$

Substituting Equations I17, I18, and I19 into Equation I20 yields

$$B'_{ij} = \frac{R}{E^* A^*} \left[ \{U\}_i^T \{N\}_j - \{U\}_j^T \{N\}_i \right] \quad [I21]$$

and substituting Equation I21 into Equation I8 yields

$$\left[ \sigma_i^2 - \sigma_j^2 \right] \int_0^\Omega \{U\}_i^T \{U\}_j d\theta = \frac{R}{E^* A^*} \left[ \{U\}_i^T \{N\}_j - \{U\}_j^T \{N\}_i \right]_0^\Omega = 0 \quad [I22]$$

The conditions under which Equation I22 is valid for  $i \neq j$  must still be determined.

Consider the elastic restraint conditions

$$\{N\} = [K] \{U\} \quad (\theta = 0) \quad [I23a]$$

$$\{N\} = -[K] \{U\} \quad (\theta = \Omega) \quad [I23b]$$

Substituting Equations I23 into Equation I22 yields

$$\left[ \sigma_i^2 - \sigma_j^2 \right] \int_0^\Omega \{U\}_i^T \{U\}_j d\theta = \frac{R}{E^* A^*} \left[ \{U\}_i^T [K] \{U\}_j - \{U\}_j^T [K] \{U\}_i \right]_0^\Omega = 0 \quad [I24]$$

Thus Equations I23 are the conditions under which Equation I9 is valid.

One last task remains to complete the orthogonality analysis. That is to find the integral of the inner product of a mode shape with itself over the closed interval  $0 \leq \theta \leq \Omega$ . When  $i = j$ , Equation I22 reduces to the indeterminate form

$$0 = \int_0^{\Omega} \{U\}_i^T \{U\}_i d\theta = 0 \quad [I25]$$

As suggested by Rayleigh (Refs. 13 and 14), L'Hospital's Rule can be used to evaluate the integral in question by considering a set of functions identical in form to  $\{U\}$  and  $\{N\}$ , except with  $\sigma$  the independent variable instead of  $\theta$ . Thus if we set

$$\sigma_j = \sigma \quad [I26]$$

$$\sigma_i = \sigma + d\sigma \quad [I27]$$

$$\frac{d(\quad)}{d\sigma} = (\quad) \quad [I28]$$

$$\{U\}_j = \{U\} \quad [I29]$$

$$\{U\}_i = \{U\} + \{\bar{U}\} d\sigma \quad [I30]$$

$$\{N\}_j = \{N\} \quad [I31]$$

$$\{N\}_i = \{N\} + \{\bar{N}\} d\sigma \quad [I32]$$

then applying L'Hospital's Rule to Equation I22 yields

$$2\sigma d\sigma \int_0^{\Omega} \{U\}^T \{U\} d\theta = \frac{R d\sigma}{E^* A^* \sigma} \left[ \{\bar{U}\}^T \{N\} - \{U\}^T \{\bar{N}\} \right]_0^{\Omega}$$

or

$$\int_0^{\Omega} \{U\}^T \{U\} d\theta = \frac{R}{2E^* A^* \sigma} \left[ \{\bar{U}\}^T \{N\} - \{U\}^T \{\bar{N}\} \right]_0^{\Omega} \quad [I33]$$

The derivatives with respect to  $\sigma$  on the RHS of Equation I33 can be evaluated as follows. All elements of  $\{U\}$  and  $\{N\}$  have the general form

$$f = f(\lambda_1 \theta, \lambda_2 \theta, \lambda_3 \theta) \quad [I34]$$

where

$$\lambda_n = \lambda_n(\sigma) \quad (n = 1, 2, 3) \quad [I35]$$

therefore

$$\frac{df}{d\sigma} = \frac{\partial f}{\partial \lambda_n} \bar{\lambda}_n \quad (\text{sum on } n) \quad [I36]$$

The derivatives with respect to  $\lambda$  are easily evaluated by the chain rule.

$$\frac{\partial f}{\partial \lambda_n} = \theta \frac{\partial f}{\partial (\lambda_n \theta)} \quad [I37]$$

The derivative of  $\lambda_n$  with respect to  $\sigma$  can be evaluated by differentiating Equation E6, which yields

$$(6\lambda^5 + 4a\lambda^3 + 2b\lambda) \bar{\lambda} + (\bar{a}\lambda^4 + \bar{b}\lambda^2 + \bar{c}) = 0 \quad [I38]$$

so that

$$\bar{\lambda}_n = - \frac{\bar{a}\lambda_n^4 + \bar{b}\lambda_n^2 + \bar{c}}{2\lambda_n [3\lambda_n^4 + 2a\lambda_n^2 + b]} \quad [I39]$$

Because of the factor  $\theta$  in Equation I37, the RHS of Equation I33 vanishes at the lower limit, and therefore Equation I33 reduces to

$$\int_0^\Omega \{U\}_i^T \{U\}_i d\theta = \Gamma_i = \frac{R}{2E^* A^* \sigma_i} \left[ \{\bar{U}\}_i^T \{N\}_i - \{U\}_i^T \{\bar{N}\}_i \right]_{\theta=0}^\Omega \quad [I40]$$

It is convenient to normalize the mode shapes so that

$$\Gamma_i = 1 \quad [I41]$$

and in that case

$$\int_0^\Omega \{U\}_i^T \{U\}_j d\theta = \delta_{ij} \quad [I42]$$

The orthogonality condition, I42, is used in Appendix J to obtain transient vibration solutions by expressing both the arch displacements and the forcing functions in modal series form. See Equations J2 and J3.

# APPENDIX J TRANSIENT ANALYSIS

Equations B17 and C7 describe the transient response of a circular arch.  
The basic equation is

$$\langle \ddot{u} \rangle - \left[ \frac{c}{R} \right]^2 \underline{L} \langle u \rangle = \frac{1}{\rho A} \langle b \rangle \quad [J1]$$

Both the displacements and the externally applied loads are assumed to be expressible in a modal series expansion of the form

$$\langle u \rangle = \sum_{j=1}^{\infty} T_j \langle U \rangle_j \quad [J2]$$

$$\langle b \rangle = \sum_{j=1}^{\infty} H_j \langle U \rangle_j \quad [J3]$$

where the mode shapes satisfy the relations

$$\underline{L} \langle U \rangle_j = -\sigma_j^2 \langle U \rangle_j = -\frac{p_j^2}{\left[ \frac{c}{R} \right]^2} \langle U \rangle_j \quad (\text{no sum}) \quad [C16]$$

and

$$\int_0^{\Omega} \langle U \rangle_i^T \langle U \rangle_j d\theta = \delta_{ij} \quad [I42]$$

Premultiplying Equation J3 by  $\langle U \rangle_i^T$  and integrating on  $\theta$  from 0 to  $\Omega$  yields

$$H_i = \int_0^{\Omega} \langle U \rangle_i^T \langle b \rangle d\theta \quad [J4]$$

Substituting Equations J2, J3, and C16 into Equation J1 yields

$$\sum_{j=1}^{\infty} \ddot{T}_j(U)_j + \sum_{j=1}^{\infty} p_j^2 T_j(U)_j = \frac{1}{\rho A} \sum_{j=1}^{\infty} H_j(U)_j \quad [J5]$$

Premultiplying Equation J5 by  $(U)_i^T$  and integrating on  $\theta$  from 0 to  $\Omega$  yields

$$\ddot{T}_i + p_i^2 T_i = \frac{1}{\rho A} H_i \quad [J6]$$

so that, assuming

$$T_i(0) = \dot{T}_i(0) = 0 \quad [J7]$$

the solution is

$$T_i(t) = \frac{1}{\rho A} \int_0^t H_i(\tau) \sin p_i(t - \tau) d\tau \quad [J8]$$

which completes the formal analysis.



# APPENDIX K VISCOELASTIC ANALYSIS

The dynamic analysis of a viscoelastic circular arch turns out to be a fairly simple extension of the results for an undamped elastic arch, when the rate-sensitive behavior is governed by a single parameter.

When the arch is viscoelastic, Equations A40 and A41 have the forms

$$\sigma_{11} = E\epsilon_1 + \alpha^* \dot{\epsilon}_1 = E \left[ \epsilon_1 + \frac{\alpha^*}{E} \dot{\epsilon}_1 \right] \quad [K1]$$

$$\sigma_{12} = G\gamma_{12} + \eta \dot{\gamma}_{12} = G \left[ \gamma_{12} + \frac{\eta}{G} \dot{\gamma}_{12} \right] \quad [K2]$$

If we assume that

$$\frac{\alpha^*}{E} = \frac{\eta}{G} = \xi \quad [K3]$$

then Equations K1 and K2 can be written in the form

$$\sigma_{11} = E (\epsilon_1 + \xi \dot{\epsilon}_1) \quad [K4]$$

$$\sigma_{12} = G (\gamma_{12} + \xi \dot{\gamma}_{12}) \quad [K5]$$

and therefore Equation B8 has the form

$$(F) = EAS (\delta) + \xi (\dot{\delta}) \quad [K6]$$

and Equations B17 and C7 have the form

$$(\ddot{u}) - \left[ \frac{c}{R} \right]^2 L \left\{ (u) + \xi (\dot{u}) \right\} = \frac{1}{\rho A} (b) \quad [K7]$$

Equation C6, which describes free vibration in a normal mode, has the form

$$\ddot{T}(U) - \left[ \frac{C}{R} \right]^2 (T + \xi \dot{T}) \underline{L}(U) = (0) \quad [K8]$$

and Equations C12 and C14 have the forms

$$\ddot{T}(m) - \left[ \frac{C}{R} \right]^2 (T + \xi \dot{T}) [\bar{U}]^{-1} \underline{L}(U) = (0) \quad [K9]$$

$$\frac{\ddot{T}}{T + \xi \dot{T}}(m) = \left[ \frac{C}{R} \right]^2 [\bar{U}]^{-1} \underline{L}(U) = -p^2(m) \quad [K10]$$

Equation C15 has the form

$$\ddot{T} + \xi p^2 \dot{T} + p^2 T = 0 \quad [K11]$$

but Equation C16 remains the same.

$$\underline{L}(U) = - \frac{p^2}{\left[ \frac{C}{R} \right]^2} (U) = -\sigma^2(U) \quad [C16 \text{ bis}]$$

Equations I14 and I16 have the forms

$$(F) = \frac{E^* A^*}{R} (T + \xi \dot{T}) (D \underline{S} - \underline{S} G^T)(U) \quad [K12]$$

$$(F) = (T + \xi \dot{T})(N) \quad [K13]$$

Equations I23 remain the same, and substituting Equations I23 into Equation K13 yields

$$(F) = (T + \xi \dot{T}) [K_J](U) = [K_J](u) + \xi(\dot{u}) \quad (\theta = 0) \quad [K14a]$$

$$(F) = -(T + \xi \dot{T}) [K_J](U) = -[K_J](u) + \xi(\dot{u}) \quad (\theta = \Omega) \quad [K14b]$$

Thus, in order for the modal orthogonality conditions to remain valid, the support restraints must also be viscoelastic, with  $\xi$  as the single rate sensitivity parameter. This will guarantee that the form of Equation I24 will be preserved.

The transient analysis problem is approached by substituting Equations J2 and J3 into Equation K7, which yields

$$\sum_{j=1}^{\infty} \ddot{T}_j(U)_j + \sum_{j=1}^{\infty} p_j^2 [T_j + \xi \dot{T}_j](U)_j = \frac{1}{\rho A} \sum_{j=1}^{\infty} H_j(U)_j \quad [K15]$$

Premultiplying Equation [K15] by  $(U)_i^T$  and integrating on  $\theta$  from 0 to  $\Omega$  yields

$$\ddot{T}_i + \xi p_i^2 \dot{T}_i + p_i^2 T_i = \frac{1}{\rho A} H_i \quad [K16]$$

The form of the convolution integral which is the solution to Equation K16 depends on the values of  $\xi$  and  $p_i$ . When  $\xi p_i < 2.0$  the modal damping is less than critical; when  $\xi p_i = 2.0$  the modal damping is critical; and when  $\xi p_i > 2.0$  the modal damping is greater than critical.

## APPENDIX L

### NUMERICAL ANALYSIS

The most difficult computational phase of the arch modal analysis is finding the roots of the frequency equation (H7). A sample problem for a Timoshenko beam was solved numerically on the AFWL CRAY. The solution involved eleven program segments, which were stored in six files. These program segments are described below. The function used was Equation 5-14 of Reference 2.

#### Program TEST

This is the main program in the solution. It performs all of the I/O, determines the interval and tolerance for root finding, and specifies the number of roots sought.

#### Function PRPCOM

This function's purpose is to prepare the common region PARAM, which contains the function parameters C12, C22, R, L, and THETA. The meanings of these variables are given below, along with the other variables of PRPCOM.

C12	$C_1^2$	square of longitudinal wavespeed
C22	$C_2^2$	square of shear wavespeed
R	r	radius of gyration
L	L	length of beam
E	E	beam modulus of elasticity
G	G	beam elastic shear modulus
RHO	$\rho$	beam density
I	I	beam moment of inertia
KPRIME	K'	shear deformation coefficient
RBIG	R	rotational beam-end restraint
THETA	$\theta$	EI/RBIG

#### Subroutine ZEROIN

This is a modified version of the ZEROIN routine found in Reference 16. The modifications were necessary to allow the routine to search for more than one root. The roots are stored in the array ROOT as they are found, and the routine continues searching until it has found the desired number of roots, or until it cannot find any more. For a discussion of the methods used to find the roots, see Reference 16.

#### Function SGN

This simple function returns 1.0 if its argument is positive, 0.0 if the argument is zero, and -1.0 if the argument is negative. This is used by ZEROIN.

#### Function G

This routine uses deflation to allow ZEROIN to find more than one root of the function. Deflation is a technique by which previously found roots are eliminated, but other roots remain. The basic idea is to find the roots of  $g(x) = f(x)/\Pi(x - r_i)$ , where  $r_i$  indicates a previously found root, and  $\Pi$  denotes a product. In this way, the roots are eliminated. Note that extreme care must be taken when evaluating the function at points close to a previous root.

#### Function FACT

This simple function returns the factorial of its integer argument. It is used by Function G.

#### Function F

This is the function whose roots are of interest. The function parameters are contained in the common region PARAM. For the trial function they were used to find XI ( $\xi$ ) and GAMMA ( $\gamma$ ), which are used to find K1 and K2. The function is then evaluated piece by piece, since it is quite complicated.

#### Function F1

This function is used to find XI and GAMMA.

#### Function F2

This function is used to find K1 and K2.

Ref. 16. Forsythe, G. E., M. A. Malcolm, and C. B. Moler, Computer Methods for Mathematical Computations, Prentice-Hall (1977).

### Function DERIV

As mentioned above, care must be taken when evaluating the function  $g(x) = f(x)/\pi(x - r_i)$  near  $x = r_i$ , a previous root. At such a point, L'Hospital's Rule must be applied. This means that the derivative of  $f(x)$  at  $x = r_i$  must be found. This function calculates the  $n^{\text{th}}$  derivative of  $f(x)$  at some point.

### A Closer Look at G and DERIV

As mentioned above,

$$g(x) \equiv f(x)/\pi(x - r_i) \quad [L1]$$

where  $f(x)$  is the function of interest and  $r_i$  is a previously found root. This definition allows us to ignore all previous roots without losing any others. A simple example will help illustrate. If

$$f(x) \equiv (x-1)(x-2)(x-3) \quad [L2]$$

then the roots are  $x = 1, 2, 3$ . Suppose that in our first search we find that  $x = 3$  is a root. Then

$$g(x) \equiv [(x-1)(x-2)(x-3)]/(x-3) = (x-1)(x-2) \quad [L3]$$

We have eliminated the root  $x = 3$ , but have retained  $x = 1$  and  $x = 2$ , so that one of them will be discovered. Note that once all the roots have been discovered,  $g(x) \equiv 1$ , so no more roots will be mistakenly discovered.

This method works equally well for functions other than polynomials, but it is more difficult to demonstrate. The problem with the method is that when ZEROIN tries to evaluate the function at a previous root, it obtains

$$g(x) \equiv f(x)/(x-r_i) = 0/0 \quad [L4]$$

which the computer cannot evaluate. However, through the application of L'Hospital's Rule, we obtain

$$g(x) \equiv f(x)/\pi(x - r_i) \equiv \left[ f'(x) / \left[ \frac{d}{dx} [\pi(x-r_i)] \right] \right]_{x=r} \quad [L5]$$

The method used to compute derivatives is quite simple. It is based on the slope interpretation of the derivative:

$$\left. \frac{df}{dx} \right|_{x=x_0} = \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} \quad [L6]$$

or

$$= \frac{f(x_1) - f(x_1 - h)}{h} \quad [L7]$$

where  $x_1 = x_0 + \Delta x$ ,  $h = 2\Delta x$ .

Higher order derivatives are found simply from lower order derivatives

$$\left. \frac{d^n f}{d^n x} \right|_{x=x_0} = \frac{\left[ \left. \frac{d^{n-1} f}{d^{n-1} x} \right|_{x=x_1} - \left. \frac{d^{n-1} f}{d^{n-1} x} \right|_{x=x_1 - h} \right]}{h} \quad [L8]$$

This procedure is implemented easily using the array DFNDNX. In the first step of the routine, enough points ( $n + 1$ ) are chosen to evaluate the  $n^{\text{th}}$  derivative. The points are evenly spaced ( $h$  is the interval width for adjacent points), and the point at which the  $n^{\text{th}}$  derivative is desired is at the midpoint of the interval. As the points are chosen, the function is evaluated at each point; and these values are stored in DFNDNX(1,i),  $i = 1, n + 1$ . In the third step, each adjacent pair of values is used to compute a single value for the derivative. This step is repeated until there is only one value. This is the  $n^{\text{th}}$  derivative.

For example, if  $f(x) = x^3 + x^2 + x$ , and the 4<sup>th</sup> derivative of  $f$  is desired at  $x = 1.0$ , then DFNDNX will contain (if  $h = 0.01$ )

DFNDNX

	1	2	3	4	5	
	$f(.98)$	$f'(0.985)$	$f''(0.99)$	$f'''(0.995)$	$f^{IV}(1.00)$	. . . .
1	2.8816	5.8807	7.9400	6.0000	0.0000	
	$f(.99)$	$f'(0.995)$	$f''(1.00)$	$f'''(1.005)$	. . . .	
2	2.9404	5.9601	8.0000	6.0000		
	$f(1.00)$	$f'(1.005)$	$f''(1.01)$	. . . .		
3	3.0000	6.0401	8.0600			
	$f(1.01)$	$f'(1.015)$	. . . .			
4	3.0604	6.1207				
	$f(1.02)$	. . . .				
5	3.1216					
	.					
	.					
	.					
	.					

These values are correct, since

$$f'(x) = 3x^2 + 2x + 1$$

$$f''(x) = 6x + 2$$

$$f'''(x) = 6$$

$$f^{IV}(x) = 0$$

Values for less well-behaved functions will have greater error.

#### Solving for Roots of Other Functions

When seeking the roots of other functions, the user should use the function  $f$  to compute the function. Any necessary parameters can be calculated or assigned in PRPCOM, and passed through the common region PARAM. The function  $f$  must be a function of one variable. The user must be aware of the meanings of the arguments of ZEROIN.

- a        the lower bound of the interval
- b        the upper bound of the interval
- f        the function of interest
- tol      the tolerance (maximum uncertainty) of the independent variable (root)



nroot      the number of roots desired  
root        the roots  
iroot      the number of roots found

Thus, in order to solve another function, both f and PRPCOM must be rewritten, as must lines 9-14 of TEST. All other code is completely compatible.

```

      real function deriv(f,n,x,h)
      implicit none
      real f,x,h
      integer n

c
      common /machin/ eps
      real eps
c
      external fndeps
c
c n cannot be greater than 10 unless array dimensions are increased
c
      real x1(11),dfndnx(11,11)
      real tol,xtemp,dfn,dnx
      integer i,j,maxj
c
      if(eps.eq.0.0)call fndeps
      tol=2.0*eps*x
      x1(1)=x-(float(n)/2.0)*h
      xtemp=x1(1)
      dfndnx(1,1)=f(xtemp)
c
      do 10 i=2,n+1
        x1(i)=x1(1)+float(i-1)*h
        xtemp=x1(i)
        dfndnx(1,i)=f(xtemp)
10  continue
c
      do 15 i=1,n+1
15  continue
c
      do 30 i=2,n+1
        maxj=n+2-i
        do 20 j=1,maxj
          dfndnx(i,j)=dfndnx(i-1,j+1)/h-dfndnx(i-1,j)/h
20  continue
30  continue
c
      do 50 i=1,n+1
50  continue
c
      deriv=dfndnx(n+1,1)
      return
      end

      subroutine fndeps
c
      common /machin/ eps
      real eps
c
      real tol
c
      eps=1.0
10  eps=eps/2.0
      tol=1.0+eps
      if(tol.gt.1.0)goto 10
c
      return
      end

```

XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX

```

real function f(p)
implicit none
real p

c
common /param/ c12,c22,r,l,theta
real c12,c22,r,l,theta

c
real f1,f2
external f1,f2

c
real xi,gamma,k1,k2
real fsign,temp
real xil,gl,col,co2,co3,co4,co5,co6,t1,t2,t3,t4

c
f=0.0
if (p.eq.0.0)goto 99
fsign=-1.0
xi=f1(fsign,p)
fsign=+1.0
gamma=f1(fsign,p)

c
fsign=-1.0
temp=xi
k1=f2(fsign,temp,p)
fsign=+1.0
temp=gamma
k2=f2(fsign,temp,p)

c
xil=xi*k1
if (xil.gt.5000.0)xil=5000.0
gl=gamma*k1

c
col=2.0
co2=2.0*theta*(gamma+xi*k2/k1)
co3=2.0*theta*(xi+gamma*k1/k2)
co4=2.0*theta*theta*gamma*xi
co5=(k2/k1)*(1.0+((theta*xi)**2))
co6=(k1/k2)*(((theta*gamma)**2)-1.0)

c
t1=col*(1-cosh(xil)*cos(gl))
t2=co2*cosh(xil)*sin(gl)
t3=co3*sinh(xil)*cos(gl)
t4=(co4+co5+co6)*sinh(xil)*sin(gl)

c
f=t1+t2-t3+t4
99
c
continue

c
return
end

real function f1(s,p)

```

11

```

|
      real s,p
c
      common /param/ c12,c22,r,l,theta
      real c12,c22,r,l,theta
c
      real t1,t2,t3,t4,t5
c
      t1=s*(1.0/c12+1.0/c22)*p*p/2.0
      t2=(1.0/c22-1.0/c12)*(p**4)/4.0
      t3=((p/r)**2)/c12
      t4=t2+t3
      t5=t1+sqrt(t4)
c
      f1=sqrt(t5)
c
      return
      end

      real function f2(s,t,p)
      real s,t,p
c
      common /param/ c12,c22,r,l,theta
      real c12,c22,r,l,theta
c
      real t1,t2,t3,t4
c
      t1=c22/(r*r)
      t2=t1-rho*rho
      t3=s*c12*t*t
      t4=t1*t
c
      f2=(t2+t3)/t4
c
      return
      end

```

\*\*\*\*\*

```

      real function g(f,x,root,nroot)
      implicit none
      real f,x
      real root(x)
      integer nroot
      external f
c
      external fndeps
      common /machin/ eps
      real eps
c
      real deriv
      integer fact
      external deriv,fact
c

```

||

```

      real tol,h
      integer i,n,nfact
      logical atroot
c
      if (eps.eq.0.0) call fndeps
      tol=abs(2.0*eps*x)
      if (tol.lt.eps) tol=eps
c
      atroot=.false.
      n=0
      do 10 i=1,nroot
         if(abs(x-root(i)).gt.tol) goto 10
         n=n+1
         atroot=.true.
10    continue
c
      nfact=fact(n)
      h=0.001
      write(7,100)nfact
100   format(' evaluating limit',i5)
      if (atroot) g=deriv(f,n,x,h)/float(nfact)
      if (.not.(atroot)) g=f(x)
c
      do 40 i=1,nroot
         if (abs(x-root(i)).lt.tol) goto 40
         write(7,200)
200   format(' deflating function')
         g=g/(x-root(i))
40    continue
c
      return
      end

      integer function fact(n)
      integer n
c
      integer i
c
      fact=1
      do 10 i=2,n
         fact=fact*i
10    continue
c
      return
      end

```

\*\*\*\*\*

```

      real function prpcom(dummy)
      implicit none
      real dummy
c

```

||

```

common /param/ c12,c22,r,l,theta
real c12,c22,r,l,theta

c
real e,g,rho,b,i,kprime,rbig
c
e=4.08e06
g=1.70e06
rho=2.247e-04
r=1.386
b=1.0
l=48.0
i=9.216
kprime=0.822467
rbig=1.0e06

c
theta=e*i/r
c12=e/rho
c22=kprime*g/rho

c
prpcom=sqrt(c22/(r*r))

c
c
return
end

```

\*\*\*\*\*

```

subroutine zeroin(ax,bx,f,tol,nroot,root,iroot)
implicit none
real ax,bx,f,tol,root(x)
integer nroot,iroot

c
external fndeps
common /machin/ eps
real eps

c
real g,sgn
external g,sgn

c
real a,b,c,d,e
real toll,xm
real fa,fb,fc
real p,q,r,s.

c
c compute eps, the relative machine precision
c
if (eps.eq.0.0)call fndeps

c initialization
c
iroot=0
a=ax

```

```

        b=bx
c
15    fa=g(f,a,root,iroot)
        fb=g(f,b,root,iroot)
c
c begin step
c
20    c=a
        fc=fa
        d=b-a
        e=d
30    if (abs(fc).ge.abs(fb)) goto 40
        a=b
        b=c
        c=a
        fa=fb
        fb=fc
        fc=fa
c
c convergence test
c
40    toll=2.0*eps*abs(b)+0.5*tol
        xm=.5*(c-b)
        if (abs(xm).le.toll) goto 90
        if (fb.eq.0.0) goto 90
c
c is bisection necessary?
c
        if (abs(e).lt.toll) goto 70
        if (abs(fa).le.abs(fb)) goto 70
c
c is quadratic interpolation possible?
c
        if (a.ne.c) goto 50
c
c linear interpolation
c
        s=fb/fa
        p=2.0*xm*s
        q=1.0-s
        goto 60
c
c inverse quadratic interpolation
c
50    q=fa/fc
        r=fb/fc
        s=fb/fa
        p=s*(2.0*xm*q*(q-r)-(b-a)*(r-1.0))
        q=(q-1.0)*(r-1.0)*(s-1.0)
c
c adjust signs
c
60    if (p.gt.0.0) q=-q
        p=abs(p)
c
c is interpolation acceptable?
c
        if ((2.0*p).ge.(3.0*xm*q-abs(toll*q))) goto 70
        if (p.ge.abs(0.5*e*q)) goto 70
        e=d

```

11

```

      d=p/q
      goto 80
c
c bisection
c
70    d=xm
      e=d
c
c complete step
c
80    a=b
      fa=fb
      if (abs(d).gt.tol1)b=b+d
      if (abs(d).le.tol1)b=b+sign(tol1,xm)
      fb=g(f,b,root,iroot)
      if ((fb*(fc/abs(fc))).gt.0.0) goto 20
      goto 30
c
c done
c
90    if (sgn(fa).ne.sgn(fb))goto 95
      if (sgn(fc).ne.sgn(fb))goto 95
      if (fb.eq.0.0)goto 95
      return
c
95    iroot=iroot+1
      root(iroot)=b
      if (iroot.eq.nroot)return
c
      a=ax
      b=bx
      goto 15
      end

      real function sgn(a)
      implicit none
      real a
      if (a.eq.0.0) goto 10
      sgn=a/abs(a)
      return
10    sgn=0.0
      return
      end

```

\*\*\*\*\*

```

program test
implicit none
real prpcom
external f,zeroin,prpcom
real a,wprime,tol,root(100),dummy
integer i,nroot,iroot

```

11



```

c      open(unit=7,file='output')
      a=0.0
      wprime=prpcom(dummy)
      tol=1.0e-6
c
      nroot=20
      call zeroin(a,wprime,f,tol,nroot,root,iroot)
      write(7,90)iroot
      do 10 i=1,iroot
        write(7,100)root(i)
      10 continue
c
      90 format(' zeroin found ', i5, ' roots')
      100 format(e12.6)
      stop
      end

```

\*\*\*\*\*

```

      zeroin found      8 roots
0.000000e+00
0.000000e+00
0.142336e+05
0.369554e+04
0.147244e+04
0.750657e+03
0.202099e+03
0.288620e+02

```

END

12-86

DTIC